

# Noncommutative obstacles of the Universe collapse. A conjecture

Petr R. Ivankov

July 19, 2008

## Abstract

New type of obstacles for the Universe collapse have been found. These obstacles appear when we replace 3D spherical model of Universe by its noncommutative approximation. Noncommutative approximation has nontrivial fundamental group that may be a cause of the obstacle. This article is a draft only. All results should be approved

## 1 Introduction

There is a set of physical models that are stable by topological reasons. Typical examples of such models are monopoles [1] or Alice strings [2]. Their stability is caused by topological invariants. Present geometrical model of the Universe space is the 3D sphere. If we replace this model by its noncommutative approximation then we can obtain new invariants. For example authors of this article had been found new invariant [3]. This invariant is trivial for 3D sphere. However it is not trivial for its noncommutative approximation. This fact may be an obstacle of the Universe collapse.

## 2 Noncommutative approximation of 3D sphere

Usual 3D sphere may be considered as a topological space. However there exists another algebraic representation where topological space is replaced by commutative algebra of continuous complex valued functions on the space [4]. There is one to one correspondence between objects of these representations. In noncommutative geometry commutative algebra is replaced by noncommutative one. Since commutative algebra is a particular case of noncommutative one usual (commutative) geometry could be considered as a case of noncommutative geometry. We would like to find a noncommutative algebra that is very similar to algebra of continuous complex valued functions defined on 3D sphere. These algebra also should provide interesting physical models. Survey of such models could be found at [5]. If noncommutative algebra is compatible with

structure of spectral triple [6] then structure of differential forms, curvature and other features of classical geometry may be constructed. Therefore this algebra may provide interesting physical model. Algebra of complex functions of 3D sphere could be generated by four real valued functions  $x_1, \dots, x_4$  those satisfy to following equations:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1. \quad (1)$$

If we introduce complex valued functions  $\alpha = x_1 + ix_2$ ,  $\beta = x_3 + ix_4$  then when we can replace (1) by the following equation:

$$\alpha\alpha^* + \beta\beta^* = 1. \quad (2)$$

Very interesting involutive noncommutative algebra is considered in [7]. It is generated by two elements  $\alpha$ ,  $\beta$  and satisfies to following relations.

$$\begin{aligned} \alpha^*\alpha + \beta^*\beta &= I, & \alpha\alpha^* + q^2\beta\beta^* &= I, \\ \alpha\beta - q\beta\alpha &= 0, & \alpha\beta^* - q\beta^*\alpha &= 0, \\ \beta^*\beta &= \beta\beta^*. \end{aligned}$$

where  $q$  is a real number and  $0 < q \leq 1$ .

This algebra is denoted by  $C(SU_q(2))$ . It is clear that if we suppose that  $q = 1$  then this algebra is commutative and it satisfies to relations (1). If  $q \approx 1$  then algebra  $C(SU_q(2))$  could be considered as noncommutative approximation of algebra  $C(S^3)$  of continuous complex valued functions on 3D sphere.  $C(SU_q(2))$  admits the structure of spectral triple[6] and noncommutative case ( $q < 1$ ) has nontrivial invariants those are trivial if  $q = 1$ .

### 3 Nontrivial invariants of noncommutative approximation of 3D sphere

Commutative geometry has a lot of invariants that belong to algebraic topology domain [8]. But noncommutative geometry has no good topology. Therefore topological invariants are replaced by algebraic ones. For example topological  $K$ -theory [9] is replaced by algebraic  $K$ -theory [10]. Authors of this article have introduced a noncommutative analogue of fundamental group. Classical geometry defines fundamental group using closed loops those looks like Fig. 1.

However noncommutative geometry has no loops and even points. Another way for definition of fundamental group is a usage of coverings [8]. The lifting of closed path on space  $X$  to covering space  $T$  looks like it is presented on Fig. 2

These liftings enable us to define a homeomorphism of covering space. And fundamental group may be defined using groups of homeomorphisms. If we would like to use this way for noncommutative geometry we shall introduce algebraic analogues of homeomorphisms and coverings. First problem was resolved

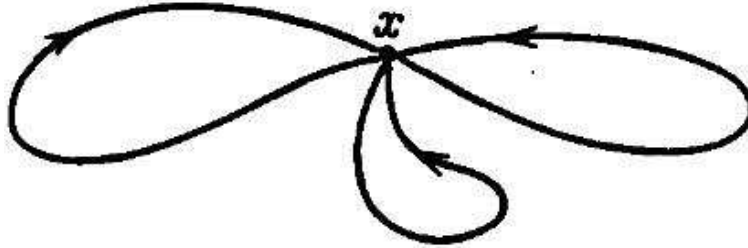


Figure 1: Closed loops.[11]

long time ago. It is known that any isomorphism corresponds to  $*$ -automorphism of algebra of continuous functions. So we can replace homeomorphisms of spaces by  $*$ -automorphisms. Second problem is resolved by authors of this article and you can find their results at [3]. The key moment a definition of unramified coverings. It is worth to recall here a difference between unramified coverings and ramified ones. Typical example of ramified covering is  $n$ -listed function of complex plane:

$$z \mapsto z^n.$$

This covering is presented on Fig. 4.

The point 0 is a point of ramification. It is clear that in this case we have ambiguous lift of path at covering space.

## 4 A sketch of construction

It is known that commutative 3D sphere has no nontrivial unramified coverings. Therefore its fundamental group is trivial. However its noncommutative approximation has such coverings. This fact has a good physical interpretation. Usual Globe sphere may be covered by another Globe where parallels wrap  $n$  times parallels. However this covering is ramified at Globe poles. Although Globe is 2D sphere the situation of 3D sphere is absolutely similar. Otherwise noncommutative geometry have no poles since noncommutative space is fuzzy. Therefore such coverings could be constructed. We refer readers to our work [3]. Soon we shall write an article about coverings of spectral triple associated with  $C(SU_q(2))$ . But this article would be overloaded by math. Here we note key features of the construction. Noncommutative geometry also operates with  $pre-C^*$  algebras those correspond to algebras of smooth complex valued function. In our case this algebra is denoted by  $\mathcal{A} = C^\infty(SU_q(2))$ . For construction of covering we should extend an algebra  $\mathcal{A}$ . It means that we shall construct such  $pre-C^*$  algebra  $\mathcal{B}$  that  $\mathcal{A} \in \mathcal{B}$  and  $\mathcal{B}$  is a finitely generated projective  $\mathcal{A}$  module [10]. One of ways to do this is to add to  $\mathcal{A}$  such element  $x$  that  $x^n \in \mathcal{A}$ .

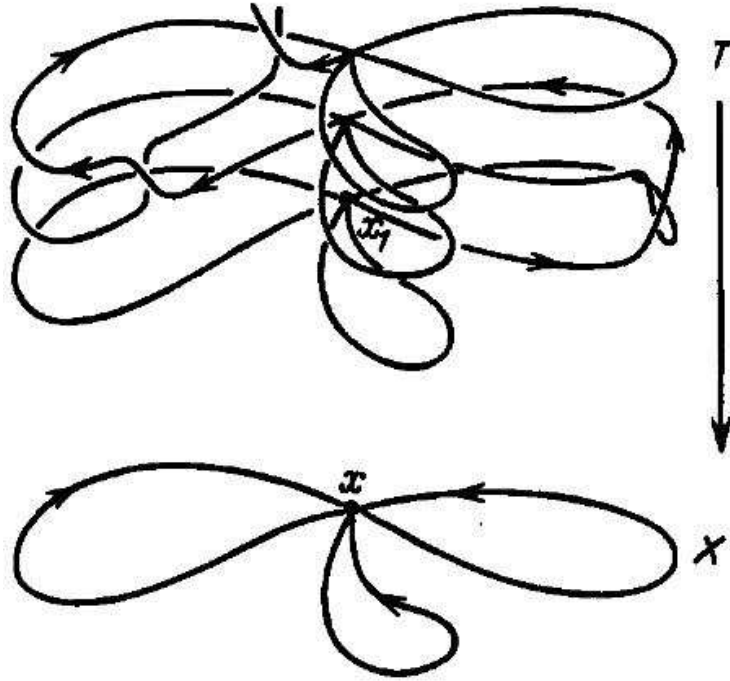


Figure 2: Lift of loop.[11]

This addition should not be arbitrary. If we select an arbitrary  $u \in C^\infty(SU_q(2))$  and suppose that  $x^n = u$  we can obtain a ramified covering. However if  $u$  is an invertible element of  $\mathcal{A}$  then the covering is unramified. However we cannot select any invertible element  $u$  for our purpose. It could happen that if for any  $n \in \mathbb{N}$  the condition  $x^n = u$  follows to  $x \in \mathcal{A}$ . It means that an addition of  $x$  do not extend algebra  $\mathcal{A}$ . So we obtain an unramified covering but this covering is trivial. If  $q < 1$  we can find such element  $u \in C^\infty(SU_q(2))$  that there does not exist such element  $x$  that  $x \in C^\infty(SU_q(2))$ ,  $x^n = u$ ,  $n \in \mathbb{N}$ ,  $n > 1$ . An article [7] contains the construction of such invertible element  $\gamma_r \in C^\infty(SU_q(2))$  that its class  $[\gamma_r] \in K_1(C^\infty(SU_q(2)))$  cannot be divided. It means that there is no such element  $a \in K_1(C^\infty(SU_q(2)))$  that  $na = [\gamma_r]$  and  $n > 1$ . If  $x^n = u$  then  $n[x] = [u]$ . It is incompatible with condition  $x \in C^\infty(SU_q(2))$ .

## 5 Construction

Spectral triple has following data set: pre  $C^*$  - Algebra Hilbert Space Dirac operator and antiisometry. Since we wish to construct odd triple the we do not need grading operator. According to [7] algebra (3) has following invertible

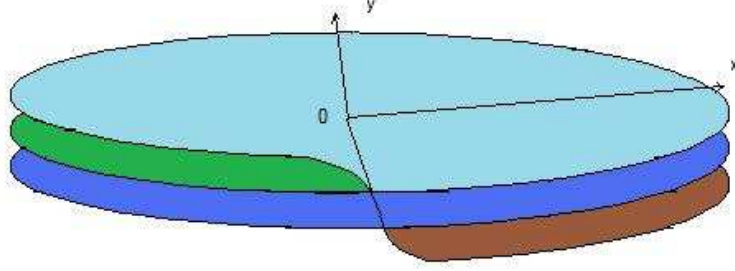


Figure 3: Ramified covering.

element

$$\gamma_r = (\beta^* \beta)^r (\beta - I) + I$$

where  $q^{2r} < \frac{1}{2} < q^{2r-2}$ .

First of all wish construct such pre  $C^*$  - algebra  $\mathcal{B}$  that  $\mathcal{A} \in \mathcal{B}$  and  $\mathcal{B}$  contains such element  $u$  that  $u^n = \gamma_r$ . In [7] pre  $C^*$  is represented by operators in Hilbert space  $\mathcal{H}$ . Let us consider vector space  $\mathcal{H}_B$  of following formal sums where  $h_0, \dots, h_{n-1} \in \mathcal{H}$ . Suppose that  $X$  is spectrum of  $\gamma_r$ . Since  $\gamma_r$  is invertible  $0 \notin X$ . So we have unramified covering of  $p : Y \rightarrow X$  given by formula  $p : z \rightarrow z^n$ . Now consider such covering  $U_i$   $i \in I$  of  $X$  by Borel subsets of  $X$  that for each  $i \in I$  preimage of  $U_i$  is disconnected set of  $n$  subsets  $V_{ij}$  ( $j = 1, \dots, n$ ) of  $Y$ . Also suppose that  $U_i \cap U_j = \emptyset$ . Since  $\gamma_r$  is normal we have spectral measure  $E$  on  $\mathcal{H}$  that corresponds to  $\gamma_r$ . Also every element  $U_i$  of covering defines orthogonal projector  $P_i$  of  $\mathcal{H}$ . Every projector  $P_i$  defines subspace  $\mathcal{H}_i \subset \mathcal{H}$  So we have following orthogonal decomposition  $\mathcal{H} = \oplus \mathcal{H}_i$ . Let be  $\mathcal{H}_B = \oplus \mathcal{H}_{ij}$  where  $\mathcal{H}_{ij} \approx \mathcal{H}_i$ . Using this isomorphism define structure of Hilbert space on  $\mathcal{H}_{ij}$ . Since  $\mathcal{H}_B$  is a direct sum of  $\mathcal{H}_{ij}$  we can provide canonical Hilbert space structure on  $\mathcal{H}_B$ . Let us define action of  $u$  on  $\mathcal{H}_{ij}$  by the following way:

$$uh = \int_{V_{ij}} z E dz$$

We will denote  $\mathcal{H}_0 \subset \mathcal{H}_B$  as subset of vectors.

$$h = \oplus h_{ij}$$

those satisfy following condition

$$h_{ij} = h_{ik}(j, k = 1, \dots, n).$$

It is clear that  $\mathcal{H}_0 \approx \mathcal{H}$ . Let  $P$  be projector on  $\mathcal{H}_0$ . So we can write:

$$h = h_0 + uh_1 + \dots + u^{n-1}h_{n-1}$$

where  $h_i = Pu^{-i}h$ . So we have following direct sum:

$$\mathcal{H}_B = \mathcal{H} \oplus u\mathcal{H} \oplus \dots \oplus u^{n-1}\mathcal{H}.$$

We will define algebra  $\mathcal{B}$  as  $C^*$  subalgebra of  $B(\mathcal{H}_B)$ . This subalgebra is generated by  $u$  and  $\mathcal{A}$ . Action of  $u$  follows from (??). Since  $\mathcal{H}$  is  $\mathcal{A}$  - module action of  $\mathcal{A}$  on the first component of (3) is also clear. We can define action of  $\mathcal{A}$  on whole  $\mathcal{H}_B$  using commutation rules between  $u$  and elements of  $\mathcal{A}$ . Since any  $C^*$  - algebra is generated by adjoint elements we can define commutation by the following way. Let  $x = \frac{u+u^*}{2}$  and  $y \in \mathcal{A}$  is self-adjoint element. Then we define  $yx \in B(\mathcal{H}_B)$  as adjoint to  $xy$  operator. Full construction of spectral triple would be appear. You can try continue full construction.

## 6 Discussion

### 6.1 Wrapping

This article is rather a short survey. We think that obstacles of the universe collapse may be contained in  $K_1(C^\infty(SU_q(2)))$ . However unrumified coverings may provide another interesting models. There are well known models of wrapping of string around cylinder. This wrapping is possible since the string has nontrivial coverings by a copy of itself. Now we can construct a wrapping of space around space-time cylinder. We can also construct models in which spaces are splitted and combined and those models look like splitting and combining of closed strings.

### 6.2 Infinitesimals

The structure of covering in noncommutative geometry provides us with a diagram of coverings  $SU_{q_m n}(2) \xrightarrow{\phi_n^m} SU_{q_m}(2)$  where  $SU_{q_n}(2)$  is a noncommutative 3D-sphere with a parameter  $q_n$ ,  $q_1$  is (arbitrary) initial number and  $\phi_n^m$  is a (class of)  $n$ -leaf covering. It seems obvious, that for any  $m, n$   $1 > q_m n > q_m$ , and  $q_{n_k} \xrightarrow{k \rightarrow \infty} 1$  for an increasing sequence  $n_k : n_{k-1} | n_k$ , although an exact dependence of  $q_n$  from  $q_1$  and  $n$  is still unclear. Taking an inverse limit of coverings, we can construct a hypothetical noncommutative 3D sphere  $SU_{q_\infty}(2)$ , where  $q_\infty$  differs from 1 by an infinitesimal[12] parameter. Still its fundamental group should stay nontrivial so an obstacle remains. Roughly saying our Universe could be infinitesimally different from commutative one, so we can't find the difference.

## References

- [1] 't Hooft-Polyakov monopole. Wikipedia
- [2] A. Schwarz: Quantum field theory and topology. Springer 1993
- [3] The Noncommutative Geometry Generalization of Fundamental Group. Petr R. Ivankov and Nickolay P. Ivankov. arXiv:math.KT/0604508, 2006
- [4] C\*-Algebras and Operator Theory, First Edition (Hardcover) by Gerald J. Murphy
- [5] Alain Connes. Noncommutative geometry and physics.
- [6] J. C. Varilly. An Introduction to Noncommutative Geometry , arXiv:physics/9709045 v 1997.
- [7] Partha Sarathi Chakraborty, Arupkumar Pal. Equivariant spectral triples on the quantum SU(2) group. arXiv:math.KT/0201004, 2003.
- [8] E.H. Spanier, Algebraic Topology, McGraw-Hill, New York, 1966.
- [9] M. Karoubi, K-theory, Springer-Verlag 1978.
- [10] H. Bass, Algebraic K-theory, Benjamin, 1968.
- [11] Fomenko A.T., Fuks D.B. Kurs gomotopicheskoy topologii, s ilustracijami, Nauka, 1989 (ru)(T).
- [12] I. S. Alverio, J.-E. Fenstad, R. Hoegh-Kron, T. Lindström, Nonstandard Methods in Stochastic Analysis and Mathematical Physics, Academic Press, New York, 1986.