Homotopy theory of C^* - algebras

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1. The Gelfand - Naimark theorem can be thought of as the construction of two contravariant functors (cofunctors for short) from the category of locally compact Hausdorff spaces to the category of C^*-algebras. The first cofunctor C takes a compact space X to the C^*-algebra $C(X)$ of continuous complex-valued functions on X, and takes a continuous map $f : X \to Y$ to its transpose $C(f) : C(Y) \to C(X)$ defined by following way: $C(f) = (h \mapsto hf);\ (h \in C(Y))$.
2. Otherwise there exists inverse functor M that sets to any commutative C^*-algebra A space of its characters $M(A)$. Many topological results related to locally compact spaces has its (noncommutative) algebraic analogues.

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Different definitions of fundamental group

1. Definition 1 Let X be topological space and $x_0 \in X$ is its point. Then fundamental group $\pi_1(X, x_0)$ as a set is a set of homotopy classes $[S^1, s_0; X, x_0]$. Since the noncommutative geometry is THE POINT IS THAT THERE IS NO POINT this definition is not suitable.
2. Definition 2 Fundamental group is a group $G(\tilde{X}|X)$ of covering transformations of universal covering $\tilde{X}$ of X.
3. Definition 1 does not have good noncommutative generalization. We need noncommutative analogue of $\tilde{X}$ for Definition 2 generalization. This problem is only partially solved.

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Hopf - Galois extensions with finite groups

If A is a ring, and G is a group which acts on A. Denote by ${}_A\mathcal{M}^G$ a category of G - equivariant modules, i.e. if $M \in_A \mathcal{M}^G$ then

$$g(am) = (ga)(gm); \; g \in G, \; a \in A, \; m \in M.$$

Natural homomorphism $A^G \to A$ is said to be (Hopf)-Galois or G - Galois extension if it satisfies following equivalent conditions:

1. The map $\mathrm{can} : A \otimes_{A^G} A \to \mathrm{Map}(G, A)$ defined as $(\sum_i a_i \otimes b_i \mapsto (g \mapsto \sum_i a_i g b_i))$ is a bijection.
2. The pair of adjoint functors

$$(-)^G :_A \mathcal{M}^G \to_{A^G} \mathcal{M},$$

$$A \otimes_{A^G} - :_{A^G} \mathcal{M} \to_A \mathcal{M}^G$$

is a pair of inverse equivalences.

Lemma

Let A be an unital algebra. Suppose that finite group G acts on A. Then following statements:

1. $\mathrm{can}_G : A \otimes_{A^G} A \to \mathrm{Map}(G, A)$ *defined as above is bijection;*
2. *There are elements* $b_i, a_i \in A$ $(i = 1, ..., n)$ *such that*

$$\sum_{i=1,\dots,n} a_i b_i = 1_A, \tag{1}$$

$$\sum_{i=1,\dots,n} a_i(gb_i) = 0 \ \forall g \in G \ (g \text{ is nontrivial}); \tag{2}$$

are equivalent.

Proof.

1. => Denote by $e \in G$ unity of G. Let $f \in \mathrm{Map}(G, A)$ be such that
$$f(e) = 1_A;\ f(g) = 0;\ (g \neq e).$$
From $A \otimes_{A^G} A \approx \mathrm{Map}(G, A)$ it follows that there are elements $a_1, ..., a_n, b_1, ..., b_n \in A$ such that $\sum_{i=1,...,n} a_i \otimes b_i$ corresponds to f i.e. $f(g) = \sum_{i=1,...,n} a_i(gb_i)$.
It is clear that elements $a_1, ..., a_n, b_1, ..., b_n$ satisfy conditions (1), (2)
2. <= Let us enumerate elements of G, i.e $G = \{g_1, ..., g_{|G|}\}$. $a_1, ..., a_n, b_1, ..., b_n$ satisfy conditions (1), (2), and let be $f \in \mathrm{Map}(G, A)$ be any map from G to A; and $x \in A \otimes_{A^G} A$ is defined as $x = \sum_{i=1,...,|G|} f(g)a_i \otimes g^{-1}b_i$. From (1), (2) it follows that $f = \mathrm{can}_G(x)$ So can_G is map onto.

Purely outer automorphisms

Definition

*Let A be C^* - algebra. A *- automorphism α is said to be* *generalized inner* *if is obtained by conjugating with unitaries from multiplier algebra $M(A)$.*

Definition

*Let A be C^* - algebra. A *- automorphism α is said to be alertpartly inner if its restriction to some non-zero α- invariant two-sided ideal is generalized inner. We call automorphism* *purely outer* *if it is not partly inner.*

Noncommutative fininte covering projection

Definition

*Let A be a C^* - algebra and $G \subset \mathrm{Aut}(A)$ is a finite subgroup of * - automorphisms. An injective * - homomorphism $f : A^G \to A$ is said to be a noncommutative finite covering projection (or noncommutative G - covering projection) if f satisfies following conditions:*

1. *A is a finitely generated equivariant G - projective left A^G Hilbert C^*-module.*
2. *If $\alpha \in G$ then α is purely outer*
3. *f is a G - Galois extension.*
4. *For any $\alpha \in \mathrm{Aut}(A^G)$ there is an automorphism $\beta \in \mathrm{Aut}(A)$ such that $\beta(a) = \alpha(a)$, $\forall a \in A^G$, i.e. $\beta|_{A^G} = \alpha$.*

G is said to be covering transformation group of f. Denote by $G(A|A^G)$ covering transformation group of covering projection

Definition

Let $A \to \mathcal{B}$ be a noncommutative finite covering projection such that:

1. *algebra B is a finite direct sum of subalgebras, i.e. $B = \bigoplus_i A_i$;*
2. *There are injective * - homomorphisms $f_i : A \to B_i$ such that $f_i(a)b = ab$, $(a \in A,\ b \in B_i)$;*
3. *Any * - homomorphisms $f_i : A \to B_i$ is a noncommutative finite covering projection*

then $A \to B$ is said to be a disconnected noncommutative finite covering projection. A noncommutative finite covering projection is said to be connected is it is not reducible.

Remark

Any C^* - algebra A has disconnected G - covering projection for any finite group G. So disconnected case is not interesing.

Let $f : A^G \to A$ be a noncommutative G - covering projection. Let $\rho : A \to \mathcal{B}(H)$ $(\rho \in \hat{A})$. Let $g \in G$ and $\rho_g : A \to \mathcal{B}(H)$ is such that $\rho_g(a) = \rho(ga)$. G acts on $\hat{A}$ $(g \mapsto (\rho \mapsto \rho_g))$; Let $g_1, ..., g_n \in G$, $n = |G|$ and $\sigma : G \times \{i, ..., n\} \to \{i, ..., n\}$ such that $\sigma(g, i) = j \Leftrightarrow g_j = gg_i$ Let $\rho_\oplus = \oplus_{g \in G} \rho_g : A \to \mathcal{B}(H^n)$ be such that

$$\rho_\oplus(a)(h_1, ..., h_n) = (\rho(g_1 a)h_1, ..., (\rho(g_n a)h_n). \qquad (3)$$

G acts on H^n such that $g(h_1, ..., h_n) = (h_{\sigma(g^{-1},1)}, ..., h_{\sigma(g^{-1},n)})$ and $g(ah) = (ga)(gh)$, i.e. $H^n \in_A \mathcal{M}^G$. Representation $\rho_\oplus$ defines representation $\eta : A^G \to \mathcal{B}(K)$. $K = (H^n)^G$. If (η, H) is not irreducible then there is a nontrivial A^G - submodule $N \subsetneq K$. From ${}_A\mathcal{M}^G \approx_{A^G} \mathcal{M}$ it follows that $A \otimes_{A^G} N \subsetneq H^n$ is a nontrivial A - submodule. If we identify H with first summand of H^n then $(A \otimes_{A^G} K) \cap H \subsetneq H$ is a nontrivial A - submodule. This fact contradicts with that (ρ, H) is irreducible. So (η, K) is an irreducible representation. In result we have $\hat{f} : \hat{A} \to \hat{B}$ $(\rho \mapsto \eta)$, $\hat{A}^G = \hat{A}/G$.

Continuous trace C^*-algebras

Let A be a C^* - algebra. For each $x \in A_+$ the (canonical) trace of $\pi(x)$ depends only on the equivalence class of an irreducible representation (π, H) of A, so that we may define a function $\hat{x} : \hat{A} \to [0, \infty]$ by $\hat{x}(t) = \mathrm{Tr}(\pi(x))$ whenever $(\pi, H) \in t$. From (Pedersen book) it follows that $\hat{x}$ is lower semicontinuous function on a in Jacobson topology.

Definition

(Pedersen book) Let A be a C^* - algebra and A_+ is its positive cone. We say that element $x \in A$ has continuous trace if $\hat{x} \in C^b(\hat{A})$. We say that C^* - algebra has continuous trace if set of elements with continuous trace is dense in A_+.

Theorem

(Pedersen book) For each C^* - algebra A there is a dense hereditary ideal $K(A)$, which is minimal among dense ideals.

Proposition

(Pedersen book) Let A be a C^* - algebra with continuous trace Then

1. *If $\pi : A \to B(H)$ is a irreducible representation then $\pi(A) = \mathcal{K}(H)$;*
2. *$\hat{A}$ is a locally compact Hausdorff space;*
3. *For each $t \in \hat{A}$ there is an abelian element such that $\hat{x} \in K(\hat{A})$ and $\hat{x}(t) = 1$.*

The last condition is sufficient for A to have continuous trace.

Proposition

Let G be a finite group and $f : A^G \to A$ is a G - covering projection. If A^G is a continuous trace C^* - algebra then A is also a continuous trace C^* - algebra

Proof.

For any irreducible representation $\rho : A \to \mathcal{B}(H)$ $\exists \eta : A^G \to \mathcal{B}(H)$

$$\rho|_{A^G} = \eta \tag{4}$$

If $x \in A^G$ is an abelian element of A^G then $\dim \eta(x) \le 1$ $\eta : A^G \to \mathcal{B}(H)$. From (4) it follows that $\dim \rho(x) \le 1$ $\rho : A \to \mathcal{B}(H)$. So if $x \in A^G$ is abelian then $x \in A$ is abelian. Let $t \in \hat{A}$ $s = \hat{f}(t) \in \hat{A}^G$. Since A^G is a continuous trace $\exists x \in A^G$, $\hat{x} \in K(\hat{A}^G)$ and $\hat{x}(s) = 1$, x is a abelian element of A, $\hat{x} \in K(A)$ and $\hat{x}(t) = \hat{x}(s) = 1$. So A is a continuous trace C^* - algebra. □

Lemma

Let G be a finite group, $f : A^G \to A$ is a noncommutative G-covering projection and A^G is separable continuous trace algebra. Then G acts freely on $\hat{A}$, there is a natural homeomorphism $\hat{A}^G \approx \hat{A}/G$ which is a topological covering projection.

Proof.

If G does not act freely on $\hat{A}$ then there are $x \in \hat{A}$ and $g \in G$ such that $gt = t$ $(t \in \hat{A})$, g should be strictly outer. if $\rho : A \to \mathcal{B}(H)$ is a representative of x then (ρ_g, H) is also representative of x. So there is unitary $U \in U(H)$ such that $\rho_g(a) = U\rho(a)U^*$ $(\forall a \in A)$. Since A is continuous trace algebra $\rho(A) = \mathcal{K}(H)$, $\rho(M(A)) = B(H)$, $\rho(U(M(A))) = U(H)$. So it is $u \in M(A)$ such that $\rho(u) = U$ and we have $\rho_g(a) = \rho(u)\rho(a)\rho(u^*)$. It means that g is inner with respect to ρ, so action of g is not strictly outer. This contradiction proves lemma. □

Let us consider Galois extensions of noncommutative torus. Noncommutative torus A_θ is C^* - norm completion of algebra generated by two unitary elements u, v and following conditions are hold:

$$uu^* = u^*u = vv^* = v^*v = 1,\ uv = e^{2\pi i\theta}vu.\ \ (\theta \in \mathbb{R})$$

Let us consider * - homomorphism $f : A_\theta \to A_{\theta'}$, where $A_{\theta'}$ is generated by unitary elements u' and v'. Homomorphism f is defined by following way:

$$u \mapsto u'^m;\ \ v \mapsto v'^n.$$

It is clear that

$$\theta' = \frac{\theta + k}{mn};\ (k = 0, ..., mn - 1).$$

Commutative C^*- subalgebras $C(u') \subset A_{\theta'}$ and $C(v') \subset A_{\theta'}$ generated by u' and v' respectively are isomorphic to algebra $C(S^1)$. There are induced by f *-homomorphisms $C(S^1) = C(u) \to C(u') = C(S^1)$, $C(S^1) = C(v) \to C(v') = C(S^1)$. These *-homomorphisms induces m and n listed covering projections respectively. Covering groups of these covering projections are $G_1 \approx \mathbb{Z}_m$ and $G_2 \approx \mathbb{Z}_n$ genrerated by $u' \mapsto e^{\frac{2\pi i}{m}} u'$; $v' \mapsto e^{\frac{2\pi i}{n}} v'$. Homomorphisms of commutative algebras $C(u) \to C(u')$, $C(v) \to C(v')$ correspond to covering projection, it follows that there are elements $x_i \in C(u')$ $(i = 1, ..., r)$, $y_j \in C(v')$ $(j = 1, ..., s)$ such that $\sum_{1 \leq i \leq r} x_i^2 = 1_{C(u')}; \sum_{1 \leq i \leq r} (g_1 x_i) x_i = 0; g_1 \in G_1; \sum_{1 \leq j \leq s} y_i^2 = 1_{C(v')}; \sum_{1 \leq j \leq s} (g_2 y_i) y_i = 0; g_2 \in G_2$ where g_1 and g_2 are nontrivial elements of G_1 and G_2. So we have a Galois extension.

Let $A_\theta \to B$ be G - Galois extension, A_θ is considered as subalgebra of B, i. e. $A_\theta \subset B$. Let $G' = \{g \in \mathrm{Aut}(B) : g|_{A_\theta} \in \mathbb{T}^2\}$. There is following exact sequence of groups:

$$\{e\} \to G \to G' \xrightarrow{f} \mathbb{T}^2 \to \{e\}. \quad (5)$$

Homomorphism f is a covering projection (in topological sense) because G is a finite group. Let us consider following special cases of sequence (5):

1. $G' = G \times \mathbb{T}^2$
2. G' is a connected topological space.

$G' = G \times \mathrm{T}^2$; $G' \approx \bigoplus_{g\in G} \mathbb{T}^2_g$; $f((t_{g_1}, ..., t_{g_n})) = t_{g_1} + ... + t_{g_n}$, $(t_{g_1}, ..., t_{g_n}) \in \bigoplus_{g\in G} \mathbb{T}^2_g$.

The $\bigoplus_{g\in G} \mathbb{T}^2_g$ is a compact Lie group and any its representation is a direct sum of irreducible representations. Any irreducible representation $\bigoplus_{g\in G} \mathbb{T}^2_g \to U(1)$ is given by:

$((z_{1g_1}, z_{2g_1}), ..., (z_{1g_n}, z_{2g_n})) \mapsto z_{1g_1}^{i_{g_1}} z_{2g_1}^{j_{g_1}} \; ... \; z_{1g_n}^{i_{g_n}} z_{2g_n}^{j_{g_n}}$; $z_{1g_k}, z_{2g_k} \in U(1)$, $(z_{1g_k}, z_{2g_k}) \in \mathbb{T}_{g_k}$, $i_{g_k}, j_{g_k} \in \mathbb{Z}$. An element $a \in B$ is said to be a *homogeneous* element of type $((i_{g_1}, j_{g_1}), ..., (i_{g_n}, j_{g_n}))$ if it satisfies following condition:

$$((z_{1g_1}, z_{2g_1}), ..., (z_{1g_n}, z_{2g_n}))a = z_{1g_1}^{i_{g_1}} z_{2g_1}^{j_{g_1}} \; ... \; z_{1g_n}^{i_{g_n}} z_{2g_n}^{j_{g_n}} a.$$

is said to be of type $((i_{g_1}, j_{g_1}), ..., (i_{g_n}, j_{g_n}))$ $(i_{g_k}, j_{g_k})))$. If a' (resp. a'') is a homogeneous element of type $((i'_{g_1}, j'_{g_1}), ..., (i'_{g_n}, j'_{g_n}))$, (resp. $((i''_{g_1}, j''_{g_1}), ..., (i''_{g_n}, j''_{g_n})))$ then the product $a'a''$ is a homogeneous element of type $((i'_{g_1} + i''_{g_1}, j'_{g_1} + j''_{g_1}), ..., (i'_{g_n} + i''_{g_n}, j'_{g_n} + j''_{g_n}))$. So B is a $\left(\mathbb{Z}^2\right)^G$ graded algebra. G naturally acts on $\left(\mathbb{Z}^2\right)^G$.

If $x \in \left(\mathbb{Z}^2\right)^G$ and $a \in B$ is homogeneous element of type x then ga is a homogeneous element of type gx. Similarly A_θ is a $\mathbb{Z}^2$ graded alebra and we for all $x \in \mathbb{Z}^2$ we can define x homogeneous elements. From exactness of $\bigoplus_{g \in G} \mathbb{T}^2_g$ action it follows that there is a nonzero homogeneous element $u_{g1} \in B$ of type $((1,0),(0,0),...,(0,0)$. Denote by u_g a homogeneous element given by: $u_g = g' u_{g_1}, \; g' g_1 = g \in G$. There is the $\mathbb{C}$ - linear map $p : B \to A_\theta$ given by $p(a) = \sum_{g \in G} ga, \forall a \in A_0$. It is clear that $p(u_{g1}) \in A_\theta$ is a $(0,1)$ homogeneous element. However any $(0,1)$ homogeneous element is equal to cu $(c \in \mathbb{C})$. If we replace u_{g_1} with $c^{-1} u_{g_1}$ then $p(u_{g_1}) = u$. From $p(u_{g1}) p(u^*_{g1}) = uu^* = 1$ it follows that

$$(u_{g_1} + ... + u_{g_n})(u^*_{g_1} + ... + u^*_{g_n}) = 1. \tag{6}$$

Right part (6) is a $((0,0),...,(0,0))$ homogeneous element. If $u_{g_1}u^*_{g_2} \neq 0$ then left part contains a nonzero summand of $((1,0),(0,-1),...,(0,0))$ type. It is impossible we have $u_{g_1}u^*_{g_2} = 0$. Similarly there are elements $v_{g_1},...,v_{g_n}$ such that $v_{g'}v^*_{g''} = 0$, $(g' \neq g'')$. If $u_{g_1}v_{g_2} \neq 0$ then right part of $uv = (u_{g1} + ... + u_{g_n})(v_{g1} + ... + v_{g_n})$ contains a nonzero homogeneous summand of $((1,0),(0,1),(0,0),...,(0,0))$ type. However uv cannot contain this summand, so $u_{g_1}v_{g_2} = 0$ since uv is homogenous of type $((1,1),...,(1,1))$, and if $g' \neq g''$ then $u_{g'}v_{g''} = 0$ Element $uu_{g_1}u^*_{g_1}$ is a sum of homogeneous elements of $((1,0),(0,0),...,(0,0))$, ..., $((0,0),(1,0),...,(0,0))$,..., $((0,0),(0,0),...,(1,0))$ However summands of $((0,0),(1,0),...,(0,0))$,..., $((0,0),(0,0),...,(1,0))$ types vanish, so we have $uu_{g_1}u^*_{g_1} = u_{g_1}$ or $ue_{g_1} = u_{g_1}$ where $e_{g_1} = u_{g_1}u^*_{g_1}$. Let $e_g = gg_1^{-1}e_{g_1}$. $\forall g \in G$ e_g is an idempotent and B is a following direct sum of algebras: $B = \bigoplus_{g\in G} e_g B$. The covering is disconnected.

G' is a connected topological space. $f : A_\theta \to A_{\theta'}$, where $A_{\theta'}$ is generated by unitary elements u' and v'. Homomorphism f is defined by following way:

$$u \mapsto u'^m;\ v \mapsto v'^n.$$

It is clear that

$$\theta' = \frac{\theta + k}{mn};\ (k = 0, ..., mn - 1).$$

If A is a C^* then $A \oplus A$ may by regarded as $\mathbb{Z}_2$ - graded algebra. If A and B are $\mathbb{Z}_2$ - graded a C^* algebras. Let $\mathbb{E}(A, B)$ is set of triples of triples (E, ϕ, F) such that E is graded Hilbert module over B, ϕ is a graded *- homomorphism from A to $\mathcal{B}(E)$, and F is an operator in $\mathcal{B}(E)$ of degree 1, such that $[F, \phi(a)]$, $(F^2 - 1)\phi(a)$ and $(F - F^*)\phi(a)$ are all in $\mathcal{K}(E)$ for all $a \in A$. Triple (E, ϕ, F) is called a $KK(A, B)$ cycle. $KK(A, B) = \mathbb{E}(A, B)/ \approx$. If $f : A_1 \to A_2$ is a graded homomorphism then for any B there is group homomorphism $f^* : KK(A_2, B) \to KK(A_1, B)$ induced by map $(E, \phi, F) \to (E, \phi \circ f, F)$. If $f : A_1 \to A_2$ is injective graded homomorphism such that A_2 is finitely generated A_1 module, (E, ϕ, F) is $KK(A_1, B)$ cycle then $(A_2 \otimes_f E, 1 \otimes \phi, 1 \otimes F)$ is $KK(A_2, B)$ cycle. So there is a homomorphism $f_! : KK(A_1, B) \to KK(A_2, B)$. It is clear that $f^* f_! \in \mathrm{End}_{KK(A_1, A_1)}(KK(A_1, B))$. So for any coneving projection $f : A \to A$" we have an invarint $f^* f_! \in \mathrm{End}_{KK(A,A)}(A, B)$.

Similarly any finite noncommutative covering projection $g : B \to B'$ supplies invariant $g^! g_* \in \mathrm{End}_{KK(B,B)}(A, B)$.

Example

Let $f' : A_\theta \to A_{\theta'}$, $f'' : A_\theta \to A_{\theta''}$ be such that

1. *$A_{\theta'}$ (resp. $A_{\theta''}$) is generated by u', v' (resp. u'', v'').*
2. *f' and f'' are defined as*

$$u \mapsto u'^m; \; v \mapsto v'^n.$$

$$u \mapsto u''^m; \; v \mapsto v'^n.$$

Then for any C^* algebra A we have:

1. $f'^* f'_! = f''^* f''_! \in \mathrm{End}_{KK(A_\theta, A_\theta)}(A_\theta, A)$
2. $f'^! f'_* = f''^! f''_* \in \mathrm{End}_{KK(A_\theta, A_\theta)}(A, A_\theta$

i. e. f' and f'' have same values of invariants.

Noncummutative case contains many coverings with same values of invariants. However $A_{\theta'} \otimes \mathcal{K} \approx A_{\theta''} \otimes \mathcal{K}$. We would like construct category such that Morita equivalent covering projections are isomorphic. So we replace any C^* - algebra A by $A \otimes \mathcal{K}$ Objects of this category are finite noncommutative finite covering projections. Let A - be a stable C^* - algebra, $f_1 : A_1^{G_1} \to A_1$, $f_2 : A_2^{G_2} \to A_2$ are noncommutative covering projections such that $A_1^{G_1} \approx A_2^{G_2} \approx A$, $k : G_2 \to G_1$ is a surjective group homomorphism. Consider *-homomorphisms $f : A_1 \to A_2$ such that

1. $f(k(g)a) = gf(a); \ \forall g \in G_2, \ \forall a \in A_1$.
2. f is $A - A$ bimodule homomorphism.

Definition

*Let $A \to A_1$, $A \to A_2$ be to finite noncommutative covering projection. An A covering morphism is a *- homomorphisms $f : A_1 \to A_2$ such that above conditions are satisfied.*

The $\pi_(-)$ is a functor from category of topological sets to category of groups. This functor is defined by following way

$$X \mapsto \pi_1(X), \ (f: X \to Y) \mapsto (\pi_1(f): \pi_1(X) \to \pi(Y))).$$

Generalization of fundamental group $\pi_1(X)$ is not known yet. However we know generalization of covering group.
Let $f: X \to Y$ be continuous map, and $\widetilde{X} \to X$, $\widetilde{Y} \to Y$ such normal coverings that following diagram

$$\begin{array}{ccc} \widetilde{X} & \longrightarrow & \widetilde{Y} \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

This diagram induces following diagram with surjective vertical arrows.

$$\begin{array}{ccc}
\pi_1(X) & \longrightarrow & \pi_1(Y) \\
\downarrow & & \downarrow \\
G(\widetilde{X}|X) & \longrightarrow & G(\widetilde{Y}|Y)
\end{array}$$

Definition

Let us consider above coverings. Homomorphism $G(\widetilde{X}|X) \to G(\widetilde{Y}|Y)$ is called Fundamental group homomorphism with respect to coverings $\widetilde{X} \to X$, $\widetilde{Y} \to Y$.

Definition

*Let A, B be C^* algebras, $f : A \to B$ * - homomorphism, $A \to \widetilde{A}$, $B \to \widetilde{B}$ noncommutative covering projections. Suppose that it is following commutative diagram:*

$$
\begin{array}{ccc}
\widetilde{A} & \xrightarrow{\widetilde{f}} & \widetilde{B} \\
\uparrow & & \uparrow \\
A & \xrightarrow{f} & B
\end{array}
$$

and homomorphism $\overline{f} : G(\widetilde{B}|B) \to G(\widetilde{A}|A)$ which satisfies following conditions

$$\widetilde{f}(g \cdot a) = \overline{f}(g) \cdot \widetilde{f}(a).$$

A homomorphism $\overline{f}$ is called Homomorphism of fundamental groups with respect to $A \to \widetilde{A}$, $B \to \widetilde{B}$ coverings.

Example

Let $A = A_\theta = \mathbb{C}[u, v]$, $B = A_{\theta/mn} = \mathbb{C}[u', v']$ *be* C^* *algebras and* $f : A \to B$ **- homomorphism defined by following way:*$u \mapsto u'^m$, $v \mapsto v'^n$. *Let* $A \to \widetilde{A}$, $B \to \widetilde{B}$ *coverings defined by following way:* $\widetilde{A} = A_{\theta/m'n'} = \mathbb{C}[\widetilde{u}, \widetilde{v}]$, $\widetilde{B} = A_{\theta/mm'nn'} = \mathbb{C}[\widetilde{u'}, \widetilde{v'}]$,, $u \mapsto \widetilde{u}^{m'}$, $v \mapsto \widetilde{v}^{n'}$, $u' \mapsto \widetilde{u'}^{mm'}$, $v' \mapsto \widetilde{v}^{nn'}$. *It is clear that* $G(\widetilde{A}, A) \approx \mathbb{Z}_{m'} \times \mathbb{Z}_{n'}$, $G(\widetilde{B}, B) \approx \mathbb{Z}_{mm'} \times \mathbb{Z}_{nn'}$. *Homomorphism* $\widetilde{f} : \widetilde{A} \to \widetilde{B}$ *is defined as:*

$$\widetilde{u} \mapsto \widetilde{u'}^{m'}, \widetilde{v} \mapsto \widetilde{v'}^{n'}.$$

Direct checking shows that Homomorphism of fundamental groups with respect to above coverings is natural surjective homomorphism:

$$\overline{f} : \mathbb{Z}_{mm'} \times \mathbb{Z}_{nn'} \to \mathbb{Z}_{m'} \times \mathbb{Z}_{n'}.$$

Let A be a C^* - algebra and suspension $\Sigma(A) = C_0(\mathbb{R}) \otimes A$. The Σ has a left adjoint in category of pro-C^*-algebras and arbitrary homomoprhisms.

Theorem

There is left adjont Ω of Σ in category of of pro-C^*-algebras. So $\mathrm{Hom}\Omega A, B) \approx \mathrm{Hom}(A, \Sigma B)$ and this bijection induces a natural bijection $[\Omega A, B] \to [A, \Sigma B]$ of homotopic classes. Σ.

Let us construct noncommutative generalization of homotopy groups π_n ($n \geq 0$)). If $X = \coprod_{\iota\in I} X_\iota$ then $\pi_0(X) \simeq \bigcup_{\iota\in I}\{X_\iota\}$ and $C(X) \simeq \oplus_\iota C(X_\iota)$. If $A = \oplus_{\iota\in I} A_\iota$ and A_ι is not a direct sum of algebras $\forall \iota \in I$ then we set

$$\pi_0(A) \simeq \bigcup_{\iota\in I}\{A_\iota\} = \bigcup \mathrm{Hom}^0_{\mathbb{C}}(\mathbb{C}, A) \qquad (7)$$

It is well known that:

$$\pi_{n+m}(X, x_0) = \pi_n(\Omega^m(X, x_0)), \qquad (8)$$

where $\Omega^m(X, x_0)$ is the m times iterated loop space. So noncommutative homotopy groups can be given as

$$\pi_n(A) = \pi_0(\Omega^n(A)), \qquad (9)$$

or

$$\pi_n(A) = \pi_1(\Omega^{n-1}(A)). \qquad (10)$$

The (10) supplies a group structute.

A discontinuous function f on space X which can be regarded as a operator $f \in B(\mathcal{L}^2(X))$. Let $A = C(X)\{f\}$ is a C^* - commutative subalgebra of $B(\mathcal{L}^2(X)$ generated by $C(X)$ and f. We have a continuous map, $\Omega(A) \to X$.

Definition

Let us represent a circle $S^1 = U(\mathbb{C})$ as $\{z \in \mathbb{C} \mid |z| = 1\}$. A natural generator of $C(S^1)$ is a unitary element $u \in U(C(S^1))$ such that u is represented by function $z \in C(S^1)$.

Definition

Let $U(\mathbb{C}) = \{z \in \mathbb{C} \mid |z| = 1\}$. A n -th root is a Borel-measurable function $\phi \in \ B_\infty(U(C(S^1))$ such that $(\phi(z))^n = z \ (\forall z \in U(\mathbb{C}))$. Let $u \in U(C(S^1))$ be a natural generator. Denote by $B(\phi)$ an operator $\phi(u)$.

Definition

We say that the unitary operator $u \in U(B(H))$ *has full spectrum if* $\mathrm{sp}(u) = \{z \in \mathbb{C} \mid |z| = 1\}$.

Definition

Let A *be a* C^* *- algebra,* $A \to B(H)$ *faithful representation and* $v \in U(B(H))$, $v^n \in A^+$, $v^i \notin A^+$ $(i < n)$. *Let* B *be a minimal* C^* *- algebra which contains following operators:*

1. $v^i a$, $(\forall i \in \mathbb{Z}, \ \forall a \in A)$;
2. $a v^i$, $(\forall i \in \mathbb{Z}, \ \forall a \in A)$;

Denote by $A\{v\} = B$. C^* *- algebra* $A \to A\{v\}$ *is said to be a* n *- th root multiplier extension.*

Example

Let A be a C^* - algebra, $u \in U(A^+)$ has full spectrum and $\nexists v \in U(A^+)$, $v^n = u$. Let $\rho : A \to B(H)$ is faitful representation, ϕ is a n - th root and $v = \phi(\rho(u))$. Then $v^n = \rho(u)$. We have a injective homomorphism $A \to A\{v\}$. Let $\alpha \in \mathrm{Aut}(A\{v\})$ be an automorphism defined by following way:

$$v \mapsto e^{\frac{2\pi i}{n}} v. \tag{11}$$

It is clear that $\alpha^n = \mathrm{Id}_{A\{v\}}$. This automorphism is said to be a natural n - cyclic automorphism. It defines action of $\mathbb{Z}_n$ on $A\{v\}$ and $(A\{v\})^{\mathbb{Z}_n} = A$. This action is said to be the natural cyclic action.

Lemma

Let A be a C^* - algebra, $u \in U(A^+)$, has full spectrum, ϕ, ψ are n -th roots, then it is isomorphism $A\{\phi(u)\} \otimes \mathcal{K} \approx A\{\psi(u)\} \otimes \mathcal{K}$

Proof.

It is clear that $A\{\phi(u)\} \otimes \mathcal{K} \approx (A \otimes \mathcal{K})\{\phi(u) \otimes 1\}$. We have following isomorphism:

$$(A \otimes \mathcal{K})\{\phi(u) \otimes 1\} \approx (A \otimes \mathcal{K})\{\psi(u) \otimes 1\},$$

$$\phi(u) \otimes 1 \leftrightarrow \psi(u) \otimes B(\phi\psi^{-1}).$$

□

Lemma

Let $f : A \to A\{v\}$ be a n -th root multiplier extension. Then f is a $\mathbb{Z}_n$ - Galois extension.

Proof.

It is clear that $u = v^n \in A$ is unitary element. Since u and v are unitary elements there are following isomorphisms:$C(u) \approx C(S^1)$; $C(v) \approx C(S^1)$. Map $C(u) \to C(v)$ corresponds to n listed covering projection $p : S^1 \to S^1$ of the circle and $\mathbb{Z}_n$ is its group of covering transformations. There are functions $a_1, ..., a_n, b_1, ..., b_n$ such that: $\sum_{i=1}^{n} a_i b_i = 1_{S^1}$; and if $g \in \mathbb{Z}_n$ is a not trivial element of covering transformation group then $\sum_{i=1}^{n} a_i g b_i = 0$. A generator of $\mathbb{Z}_n$ acts on $A[v]$ by following way $v \mapsto e^{\frac{2\pi i}{n}} v$. □

Example

Let A be a C^* - algebra, $x \in K^1(A)$ a generator of infinite order, $u \in U((A \otimes \mathcal{K})^+)$ is representative of x, i.e. $x = [u]$. If $\phi \in B_\infty(U(\mathbb{C}))$ is an n - th root, then $\phi(u) \notin (A \otimes \mathcal{K})^+$. So $A \otimes \mathcal{K} \to (A \otimes \mathcal{K})\{\phi(u)\}$ is a $\mathbb{Z}_n$ - Galois extension.

We assume that all coverings of above coverings are two listed coverings. Symbol $S^1_{2^n} \to S^1$ means that that covering degree of initial circle S^1 is equal to 2^n. It is natural bijection between $C(S^1)$ and $\{f :\in C[-2^n\pi, 2^n\pi], f(-2^n\pi) = f(2^n\pi)]\}$. Any $f \in C_0(\mathbb{R})$ can be regarded as limit of functions supported on intervals $[-2^n\pi, 2^n\pi]$, $n \to \infty$:

$$f_{2^n}(x) = \begin{cases} f(x) - f(2^n\pi) + (x - 2^n\pi)\frac{f(x+2^n\pi)-f(x-2^n\pi)}{2^{n+1}\pi}; \; x \in [-2^n\pi, 2^n\pi \\ 0; \; x \notin [-2^n\pi, 2^n\pi] \end{cases} \quad (12)$$

f_{2^n} is norm convergent to f. Now let us note that $L^2(\mathbb{R}) = \hat{\bigoplus}_{k\in\mathbb{Z}} L^2([2\pi k, 2\pi(k+1)])$, Otherwise f_{2^n} can be naturally identified with element $\overline{f}_{2^n} \in S^1_{2^n}$. So any $f \in C_0(\mathbb{R}) \subset B(L^2(\mathbb{R})$ can be "approximated" by by operators $f_{2^n} \in S^1_{2^n}$

Let A and $u \in U(A)$ satisfies to conditions of above definition Algebra A faithfully acts on Hilbert space H. Let us consider following sequence of multiplier extensions (See 13) :

$$A \to A\{v_2\} \to A\{v_4\} \to ... \to A\{v_{2^n}\} \to ... \quad (13)$$

where $v_{2^n}^{2^n} = u$.

Let us define action of $C_0(\mathbb{R})$ on Hilbert sum $\widetilde{H} = \bigoplus_{n\in\mathbb{Z}} H_n$. First of all note that $C(v_{2^k})$ acts on $\bigoplus_{-2^k\leq n<2^k-1} H_n$. Suppose that $C(v_{2^k})$ acts trivially on H_n if $n < -2^k \bigvee n \geq 2^k$. This action can be continued whole sum $\widetilde{H}$. Let $f \in C(\mathbb{R})$ any function and sequence f_{2^n} is defined by equation 12. On can define functions $\overline{f}_{2^n} \in C(S^1_{2^n}) = C(v_{2^n})$. So I f_{2^n} $\forall n \in \mathbb{N}$ defines bounded operator $B(f_{2^n}) \in B(\widetilde{H})$. Sequence $B(f_{2^n})$ is norm convergent. By $B(f)$ denote its limit.

Denote $B(a) \in B(\widetilde{H})$ $a \in A$ bounded operator which acts on every component of Hilbert sum as well as a acts on H.

Definition

Let $B \in B(\widetilde{H})$ be norm completion of algebra generated by elements $B(a)B(f)$ and $B(f)B(a)$, where $a \in A$ and $f \in C(\mathbb{R})$. This algebra is called Noncommutative generalization of $\mathbb{R} \to S^1$ covering.

Remark

Suppose that there are two elements $u_1, u_2 \in A$ which satisfy above conditions. We can define two actions of $C(\mathbb{R})$ on $\widetilde{H}$. Action of $C(\mathbb{R})_1$, $C(\mathbb{R})_2$ is constructed by usage of elements u_1 and u_2 respectively. Norm completion of algebra generated by $B(f_1)B(f_2)B(a)$,$B(f_2)B(f_1)B(a)$, $B(f_1)B(a)B(f_2)$, $B(f_2)B(a)B(f_1)$, $B(a)B(f_1)B(f_2)$, $B(a)B(f_2)B(f_1)$, where $a \in A$, $f_1 \in C(\mathbb{R})_1$ $f_2 \in C(\mathbb{R})_2$, is generalization of covering of torus by plane. Similarly generaization of $\mathbb{R}^n \to \mathbb{T}^n$ can be constructed.

Notion of Hurewicz homomorphism was initially appeared in algebraic topology and then generalized in several directions. This chapter is devoted to generalization related to theory of C^* - algebras. First of all let us remind some notions of algebraic topology. Let X be topological space, $x_0 \in X$ is base point, $\pi_n(X, x_0)$, $H_n(X)$ are n-th ($n \in \mathbb{N}$) are n-th homotopy group and singular homology group respectively. Then $\forall n \in \mathbb{N}$ there is natural homomorphism $\phi_n : \pi_n(X, x_0) \to H_n(X)$. This homomorphism is named *Hurewicz homomorphism*. This notion can be naturally extended for any generalized homology theory $h_*(-)$ (See [**?**]) such that $h^*(S^n) \approx \mathbb{Z}$. If K^1 is K - homology theory of C^* algebras then $K_{n \mod 2}(C(S^n)) \approx \mathbb{Z}$. So we have natural transformation $\pi_*(-) \to K^*(C(-))$ which can be regarded as generalization of Hurewicz homomorphism. In case of fundamental group we have a natural homomorphism

$$\pi_1(X) \to K^1(C(X)), \ [f] \mapsto K^1(f)(u) \tag{14}$$

rewritten by following way

$$\mathrm{Ext}^1(K_0(A), \mathbb{Z}) \to K^1(A) \to \mathrm{Hom}(K_1(A), \mathbb{Z})). \tag{15}$$

If G is an abelian group that

$$\mathrm{Ext}^1(G, \mathbb{Z}) = \mathrm{Ext}^1(G_{tors}, \mathbb{Z})$$

$$\mathrm{Hom}(G, \mathbb{Z}) = \mathrm{Hom}(G/G_{tors}, \mathbb{Z})).$$

$K^1(A)$ depends on $K_0(A)_{tors}$ and $K_1(A)/K_1(A)_{tors}$. We say that dependence on $K_0(A)_{tors}$ is a *torsion special case* and dependence on $K_1(A)/K_1(A)_{tors}$ is a *free special case*.

Example

It is well known that $\pi_1(S^1) \approx \mathbb{Z}$ and $K^1(C(S^1)) \approx \mathbb{Z}$. Natural homomorphism is an isomorphism. Let $u \in U(C(S^1))$ is such that $[u] \in K_1(S^1)$ is a generator of $K_1(S^1)$. Let $C(S^1) \to B(H)$ be a full represetation and ϕ is an n - th root. If $v = \phi(u) \in B(H)$ then $v^n = u$ and $v \notin C(S^1)$. So we have n - th root multiplier extension $C(S^1) \to C(S^1)\{v\}$.

Let A be a C^* - algebra such that $K^1(A) \approx G \oplus \mathbb{Z}$. From (15) it follows that $K_1(A) = G' \oplus \mathbb{Z}$. Let $u \in U((A \otimes \mathcal{K})^+)$ is such that $[u] \in K_1(A)$ is a generator of second summand of $G' \oplus \mathbb{Z}$. From it follows that there is $u \in U(A)$ such that $[u] \in K_1(A)$ is element of infinite order and $[u]$ is not divisible. Suppose that u has full spectrum (See 12). Let $A \to B(H)$ and $v \in U(B(H))$ is such that $v^n = u$ $(n > 1)$ then we can construct noncommutative finite covering projections $A \to A\{v\}$.

Example

It is known that $S^3 \approx SU(2)$ *as spaces,* $K_1(C(SU(2))) \approx \mathbb{Z}$ *and* $K_1(C(SU(2)))$ *is generated by unitary* $u \in U(C(SU(2) \otimes \mathbb{M}_2(\mathbb{C}))$. *Element* u *can be regarded as the natural map* $SU(2) \to \mathbb{M}_2(\mathbb{C})$ *and* u *has full spectrum. Let* ϕ *be a* n *- th root, and* $v = \phi(u)$. *Let* $A = C(SU(2)) \otimes \mathbb{M}_2(\mathbb{C})$. *Both* A *and* $A\{v\}$ *are continuous trace algebras. Let* $\rho \in \mathrm{Irr}(A\{v\})$ *be a irreducible representation of* B. *Then* $V = \rho(v) \in U(\mathbb{M}_2(\mathbb{C}))$ *unitary matrix. The unordered pair of eigenvalues* λ_1, λ_2 *of* V *is an invariant of* ρ, *i.e. unitary equivalent representations have equal eigenvalues. Group* Z_n *acts on the invariant:* $\lambda_1 \mapsto c\lambda_1, \ \lambda_2 \mapsto c\lambda_2, \ c = e^{\frac{2\pi k}{n}}, \ k \in \mathbb{N}$. *Group* Z_n *acts freely on this invariant and therefore* Z_n *acts freely on* $\widehat{A\{v\}}$. *So* $\widehat{A\{v\}} \to \hat{A}$ *is a* n *- listed covering projection. Since* $\pi_1(SU(2)) = \pi_1(S^3)$ *is trivial* $\hat{B} = \amalg_{i=1,\dots,n} S^3$ *is a disjoint union of* n *copies of* S^3. *We have a disconnected covering projection.*

Remark

Addition of discontinuous function can transform connected space to disconnected one. Topological fundamental group concerns with connected covering projections. In example 5 we have connected covering space, in example 6 we have disconnected covering space.

Example

Let A_θ be a noncommutative torus, $K^1(A_\theta \approx \mathbb{Z}^2)$ Let $u, v \in U(A)$ be reprezentatives of generators of $K^1(A_\theta)$ Both u and v has full spectrum. We can construct generalization of universal covering projection.

Example

Coverings of noncommutative 3D Sphere Algebra of complex functions of 3D sphere could be generated by four real valued functions x_1,...,x_4 those satisfy to following equations:

Example

Let $f: S^1 \to S^1$ be a n listed covering projection of the circle, C_f is the (topological) mapping cone of f. $C(f): C(S^1) \to C(S^1)$ is defined as $u \mapsto u^n$. Algebraic mapping cone $C_{C(f)} \approx C(C_f)$ of $C(f)$ corresponds to the topological space C_f. $C_{C(f)}$ is an algebra of continuous maps $f[0,1) \to U(\mathbb{C})$ such that:

$$f(0) = \sum_{k \in \mathbb{Z}} a_k u^{kn}, \ a_k \in \mathbb{C}.$$

$\pi_1(C_f) \approx \mathbb{Z}_n$ and $K^0(C_f) = K_0(C(C_f)) \approx K^1(C(C_f)) \approx \mathbb{Z}_n$. A map $v = (x \mapsto u)$ $(\forall x \in [0,1])$ has full spectrum and v but $v \notin (C(C_f))^+$. However $v^n \in (C_{C(f)})^+$. Let $C_{C(f)}) \to B(H)$ be a faithful repersentation. Homomorphism $C_{C(f)} \to C_{C(f)}\{v\}$ is a $\mathbb{Z}_n$ - Galois extension. It corresponds to (connected) n - listed universal topological covering projection of C_f.

Let A be a C^* - algebra $K_0(A) = G \oplus \mathbb{Z}_n$, where G is an abelian group. From (15) it follows that $K^1(A) \approx G' \oplus \mathbb{Z}_n$ Let $Q^s(A) = (M(A \otimes \mathcal{K})/(A \otimes \mathcal{K})$ be the stable multiplier algebra of C^* - algebra A. $K_1(Q^s(A)) = K_0(A)$ Let $x \in K^1(Q^s(A))$ be a generator of direct summand $\mathbb{Z}_n \subset K^1(Q^s(A))$ and $u \in U(Q^s(A))$ is representative of x, i.e $x = [u]$. Suppose that u has full spectrum. Let ϕ be a n - th root such that $\phi(u^n) = u$. Let $p : (A \otimes \mathcal{K})^+ \to U((A \otimes \mathcal{K})^+)/(A \otimes \mathcal{K})$ be a natural surjective *-homomorphism. Since $n[u] = [u^n] = 0$ then there is an unitary $v \in U(M(A \otimes \mathcal{K}))$ such that $p(v) = u^n$. Suppose that v has full spectrum. Now we can construct multiplier extension $A \otimes \mathcal{K} \to (A \otimes \mathcal{K})\{\phi(v)\}$.