Homotopy theory of C^* - algebras

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Abstract

Gelfand - Naĭmark theorem supplies contravariant functor from category of commutative C^*- algebras to category of locally compact Hausdorff spaces. Therefore C^*- algebra is alternative representation of a topological space. Similarly category of (noncommutative) C^*- algebras may be regarded as category of generalized (noncommutative) locally compact Hausdorff spaces. Generalizations of topological invariants may be defined by algebraic methods. For example Serre Swan theorem [30] states that complex topological K - theory coincides with K - theory of C^* - algebras. However algebraic topology have rich set of invariants. Some invariants do not have noncommutative generalizations. This article contains several steps towards definition of noncommutative generalization of homotopy groups.

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2.1. Following Gelfand-Naǐmark theorem [33] states that category of locally compact Hausdorff topological spaces is equivalent to category of commutative C^*- algebras.

Theorem 2.2. *Let* **Haus** *be the category of locally compact Hausdorff spaces with continuous proper maps as morphisms. And, let* $\mathbf{C^*Comm}$ *be the category of commutative C -algebras with proper *-homomorphisms (send approximate units into approximate units) as morphisms. There is a contravariant functor* $C : \mathbf{Haus} \to \mathbf{C^*Comm}$ *which sends each locally compact Hausdorff space X to the commutative* C^* *-algebra* $C_0(X)$ $(C(X)$ *if X is compact). Conversely, there is a contravariant functor* $\Omega : \mathbf{C^*Comm} \to \mathbf{Haus}$ *which sends each commutative* C^* *-algebra A to the space of characters on A (with the Gelfand topology).*
The functors C and Ω *are an equivalence of categories.*

So any (noncommutative) C^*- algebra may be regarded as generalized (noncommutative) locally compact Hausdorff topological space. We may summarize several properties of the Gelfand Naǐmark cofunctor with the following dictionary.

TOPOLOGY	ALGEBRA
Locally compact space	C^* - algebra
Compact space	Unital C^* - algebra
Continuous map	*-homomorpfism
Minimal compactification	Unitization
Maximal compactification	Algebra if multpicators
Closed subset	Ideal
Pointed space (X, x_0)	Pointed algebra (A, α) (See 13.22)
Disjoint union of topological spaces ($\coprod X_\iota$)	Direct sum of (pro) - C^* algebras ($\oplus A_\iota$).
Morphism of covering	?
Fundamental group	?
Singular homology	?
Hurewicz homomorphism	?

Involutive homomorphism in $\mathbf{C^*Comm}$ is just continuous map. However $*$ - homomorphism is not always good notion of morphism in category of (noncommutative) C^* - algebras (See [36], [35]). Morita equivalence and Kasparov intersection product [28] are also morphisms of C^* - algebras. Isomorphisms may be regarded as particular case of Morita equivalence and every *- homomorphism defines unique Kasparov intersection product. If two commutative C^*- algebras are Morita equivalent then they are isomorphic. So

Morita equivalence does not yield substantially new results for commutative case. Morita equivalent rings have isomorphic centers. This implies that Morita equivalent Abelian rings are already isomorphic. Thus Morita equivalence is essentially a non-commutative phenomenon. This gives another reason why so many homology functors are Morita invariant: Usually, they arise as extensions of functors defined on a category of commutative algebras to a category of noncommutative algebras. But Morita invariance imposes no restrictions whatsoever on functors defined on a category of commutative algebras, so that we can hope for a Morita invariant extension. Examples show that if a functor can be extended “naturally”, then the extension tends to be indeed Morita invariant. Morita equivalent rings also have equivalent categories of right modules and bimodules. It is also easy to see that they have equivalent lattices of ideals, so that the properties of being Noetherian, Artinian, or simple are Morita invariant (cf. [10]). They have isomorphic categories of projective modules and thus equivalent K-theories. More generally, a decent (co)homology theory should be Morita invariant, and this is indeed true for cyclic homology, Hochschild homology. Moreover, Morita equivalent rings have isomorphic centers. This implies that Morita equivalent Abelian rings are already isomorphic. Thus Morita equivalence is essentially a non-commutative phenomenon. This gives another reason why so many homology functors are Morita invariant: Usually, they arise as extensions of functors defined on a category of commutative algebras to a category of noncommutative algebras. But Morita invariance imposes no restrictions whatsoever on functors defined on a category of commutative algebras, so that we can hope for a Morita invariant extension. Examples show that if a functor can be extended “naturally”, then the extension tends to be indeed Morita invariant. A generalization of *- homomorphism will be used below for definition of fundamental group. This generalization uses some ideas related to Morita equivalence.
This article assumes elementary knowledge of following subjects.

1. Set theory [38].
2. Category theory [34], [1].
3. Algebraic topology [34], [1].
4. C^*- algebras and operator theory [28], [29], [33], [47].

The terms "set", "family" and "collection" are synonyms, and the term "class" is reserved for an aggregate which is not assumed to be a set [38]. A category [40] is said to be *small* if whose class of objects is a set. Following table contains used in this paper notations.
The set of primitive ideals is a topological space with the hull-kernel topology (or Jacobson topology).

Symbol	Meaning
$\mathbb{N}$	monoid of natural numbers
$\mathbb{Z}$	ring of integers
$\mathbb{Z}_m$	ring of integers modulo m
$\mathbb{R}$ (resp. $\mathbb{C}$)	Field of real (resp. complex) numbers
$\mathbb{C}^* = \mathbb{C} - \{0\}$	Multiplicative group of complex numbers
H	Hilbert space
$I = [0,1] \subset \mathbb{R}$	Closed unit interval
A''	Bicommutant of C^* algebra A [46]
$\mathcal{B}(H)$	Algebra of bounded operators on Hilbert space H
$\mathcal{K}(H)$ or $\mathcal{K}$	Algebra of compact operators on Hilbert space H
$U(H) \subset \mathcal{B}(H)$	Group of unitary operators on Hilbert space H
$U(n) = U(\mathbb{C}^n)$	Group of unitary operators on Hilbert space $\mathbb{C}^n$
$U(A) \in A$	Group of unitary operators of algebra A
$M(A)$	Multiplier algebra of C^*- algebra A
$Q(A) = M(A)/A$	Outer multiplier algebra of C^*- algebra A
$M^s(A) = M(A \otimes \mathcal{K})$	Stable multiplier algebra of C^*- algebra A
$Q^s(A) = (M(A \otimes \mathcal{K}))/(A \otimes \mathcal{K})$	Stable outer multiplier algebra of C^*- algebra A
$\mathbb{M}_n(A)$	The $n \times n$ matrix algebra over C^*- algebra A
A^+	C^*- algebra A with adjointed identity
A_+	Positive cone of C^*- algebra A
$\mathrm{Aut}(A)$	Group * - automorphisms of C^* algebra A
(π, H)	Representation of C^* algebra A, i.e. * - homomorphism $A \to B(H)$
$\mathbb{C}^* = U(1) = \{z \in \mathbb{C} \mid \|z\| = 1\}$	Group of unitary elements in $\mathbb{C}$
$\mathbb{C}_1 = \mathbb{C} \oplus \mathbb{C}$	Complex Clifford algebra with standard odd grading [28]
$C(X)$	C^* - algebra of continuous complex valued functions on topological space X
$C^b(X)$	C^* - algebra of bounded continuous complex valued functions on topological space X
$\hat{A}$	Spectrum of C^* - algebra A with the hull-kernel topology (or Jacobson topology)
$\Omega : \mathbf{C^*Comm} \to \mathbf{Haus}$	Natural cofunctor from category of commutative C^* algebras, to category of Hausdorff spaces
$G_{tors} \subset G$	The torsion subgroup of an abelian group

If A is a ring, and G is a group which acts on A. Denote by ${}_A\mathcal{M}^G$ a category of G - equivariant modules, i.e. if $M \in_A \mathcal{M}^G$ then

$$g(am) = (ga)(gm);\ g \in G,\ a \in A,\ m \in M.$$

Definition 2.3. [19]. A *Hermitian module* over a C^*-algebra A is the Hilbert space H of a non-degenerate *-representation $\pi : A \to B(H)$, together with the action $a \cdot \xi = \pi(a)\xi$ for $a \in A$, $\xi \in H$.

Let G be a group and A is a C^* - algebra. Denote by Herm_A (resp. Herm_A^G) category of Hermitian A - modules (resp. $A - G$ modules).

Theorem 2.4. *[39] If A is a C^* - algebra then there is a natural equivalence of categories* $\mathrm{Herm}(A) \approx \mathrm{Herm}(A \otimes \mathcal{K})$.

3 Fundamental groupoid and fundamental group

3.1 Fundamental groupoid/group in algebraic topology

Algebraic topology concern with good spaces as well as Hausdorff, (locally) compact or cellular spaces etc. Paths and their homotopies yield satisfactory definition of fundamental groupoid/group for these spaces. Let us recall necessary algebraic topology definitions. A *groupoid* is a small category in which every morphism is equivalence Paths and their homotopies can be used in case of good spaces.

Theorem 3.1. *[34] For each topological space X there is a category $\mathcal{P}(X)$ whose objects are points of X, whose morphisms from x_0 to x_1 are path classes with x_0 as origin and x_1 as end, and whose composite is product of path classes.*

The category $\mathcal{P}(X)$ is called *category of path classes* of X or the *fundamental groupoid* of X because of following theorem.

Theorem 3.2. *[34] $\mathcal{P}(X)$ is a groupoid.*

Definition 3.3. Let X be a topological space and let $x_0 \in X$. The *fundamental group of X based at x_0*, denoted by $\pi(X, x_0)$ is defined to be the group of path classes with x_0 as origin and end.

If topological space X is locally path connected and semilocally 1-path connected [34] then exist universal covering projection $p : \tilde{X} \to X$. In this case fundamental group is group of covering transformations $G(\tilde{X}|X)$ of universal covering projection p.

4 Noncommutative covering projections

4.1 Coaction of Hopf algebras

Let H be a Hopf algebra over a commutative ring $\mathbb{C}$, with bijective antipode S. We use the Sweedler notation [48] for the comultiplication on $H : \Delta(h) = h_{(1)} \otimes h_{(2)}$. $\mathcal{M}^H$ (respectively ${}^H\mathcal{M}$) is the category of right (respectively left) H-comodules. For a right H-coaction ρ (respectively a left H-coaction λ) on a $\mathbb{C}$ -module M, we denote

$$\rho(m) = m_{[0]} \otimes m_{[1]};\ \lambda(m) = m_{[1]} \otimes m_{[0]}.$$

The submodule of coinvariants $M^{\mathrm{co}H}$ of a right (respectively left) H-comodule M consists of the elements $m \in M$ satisfying

$$\rho(m) = m \otimes 1 \tag{1}$$

respectively

$$\lambda(m) = 1 \otimes m. \tag{2}$$

Definition 4.1. Let A be associative algebra and $A \in \mathcal{M}^H$. Algebra A is said to be H - *comodule algebra* if H - coaction $\rho : A \to A \otimes H$ satisfies following conditions:

$$\rho(ab) = a_{[0]}b_{[0]} \otimes a_{[1]}b_{[1]};\ \forall a, b \in A; \tag{3}$$

$$a \otimes \Delta(h) = \rho(a) \otimes h. \tag{4}$$

Let A be a right H-comodule algebra.${}_A\mathcal{M}^H$ and $\mathcal{M}_A^H$ are the categories of left and right relative Hopf modules. We have two pairs of adjoint functors ($F_1 = A \otimes_{A^{\mathrm{co}H}} -, G_1 = (-)^{\mathrm{co}H}$) and ($F_2 = \otimes_{A^{\mathrm{co}H}} A, G_2 = (-)^{\mathrm{co}H}$) between the categories ${}_{A^{\mathrm{co}H}}M$ and ${}_A\mathcal{M}^H$, and between $\mathcal{M}_{A^{\mathrm{co}H}}$ and $\mathcal{M}_A^H$. The unit and counit of the adjunction $(F_1, G_1)$are given by the formulas

$$\eta_{1,N} : N \to (A \otimes_{A^{\mathrm{co}H}} N)^{\mathrm{co}H},\ \eta_{1,N}(n) = 1 \otimes n;$$

$$\varepsilon_{1,M} : A \otimes_{A^{\mathrm{co}H}} M^{\mathrm{co}H} \to M,\ \varepsilon_{1,M}(a \otimes m) = am.$$

The formulas for the unit and counit of (F_2, G_2) are similar. Consider the canonical maps

$$\mathrm{can} : A \otimes_{A^{\mathrm{co}H}} A \to A \otimes H,\ \mathrm{can}(a \otimes b) = ab_{[0]} \otimes b_{[1]}; \tag{5}$$

$$\mathrm{can}' : A \otimes_{A^{\mathrm{co}H}} A \to A \otimes H,\ \mathrm{can}'(a \otimes b) = a_{[0]}b \otimes a_{[1]}. \tag{6}$$

Theorem 4.2. *[22] Let A be a right H-comodule algebra. Consider the following statements:*

1. *(F_2, G_2) is a pair of inverse equivalences;*
2. *(F_2, G_2) is a pair of inverse equivalences and $A \in_{A^{\mathrm{co}H}} \mathcal{M}$ is flat;*
3. can *is an isomorphism and $A \in_{A^{\mathrm{co}H}} \mathcal{M}$ is faithfully flat;*
4. *(F_1, G_1) is a pair of inverse equivalences;*
5. *(F_1, G_1) is a pair of inverse equivalences and $A \in \mathcal{M}_{A^{\mathrm{co}H}}$ is flat;*
6. can$'$ *is an isomorphism and $A \in \mathcal{M}_{A^{\mathrm{co}H}}$ is faithfully flat.*

These the six conditions are equivalent.

Definition 4.3. If conditions of theorem 4.1 are hold, then A is said to be *left faithfully flat H-Galois extension*

It is well-known that can is can isomorphism if and only if can$'$ is an isomorphism.

4.2 Action of finite group

Let G be a finite group. A set $H = \text{Map}(G, \mathbb{C})$ has a natural structure of Hopf algebra (See [26]). Addition (resp. multiplication) on H is pointwise addition (resp. pointwise multiplication).. Let $\delta_g \in H, (g \in G)$ be such that

$$\delta_g(g') \begin{cases} 1 & g' = g \\ 0 & g' \neq g \end{cases} \tag{7}$$

Comultiplication $\Delta : H \to H \otimes H$ is induced by group multiplication

$$\Delta f(g) = \sum_{g_1 g_2 = g} f(g_1) \otimes f(g_2); \ \forall f \in \text{Map}(G, \mathbb{C}), \ \forall g \in G.$$

i.e.

$$\Delta \delta_g = \sum_{g_1 g_2 = g} \delta_{g_1} \otimes \delta_{g_2}; \ \forall g \in G.$$

Action $G \times A \to A, \ (g, a) \mapsto ga$ naturally induces coaction $A \to A \otimes H$ ($H = \text{Map}(G, \mathbb{C})$) .

$$a \mapsto \sum_{g \in G} ga \otimes \delta_g \tag{8}$$

Equations (3), (4) are equivalent to following conditions of group action

$$g(a_1 a_2) = (ga_1)(ga_2), \ \forall g \in G, \ a_1, a_2 \in A,$$

$$(g_1 g_2)a = g_1(g_2 a), \ \forall g_1, g_2 \in G, \ a \in A.$$

Any element $x \in A \otimes H$ can be represened as following sum

$$x = \left(\sum_{g \in G} a_g \otimes \delta_g \right).$$

Let $a \in A$ be such that $ga = a, \ \forall g \in G$ then

$$a \mapsto \sum_{g \in G} a \otimes \delta_g = a \otimes 1. \tag{9}$$

From (9) it follows that $A^{\text{co}H} = A^G$, where $A^G = \{a \in A : ga = a; \ \forall g \in G\}$ is an algebra of invariants. There is a bijective natural map

$$A \otimes H \xrightarrow{\approx} \text{Map}(G, A) \tag{10}$$

$$\sum_{g \in G} a_g \otimes \delta_g \mapsto (g \mapsto a_g).$$

From 9 it follows that 5) can be represented in terms of group action by following way

$$\operatorname{can}\left(\sum_{i=1,\dots,n} a_i \otimes b_i\right) = \sum_{\substack{i=1,\dots,n\\ g\in G}} a_i(gb_i)\otimes\delta_g. \tag{11}$$

There is the unique map $\operatorname{can}^G : A \otimes_{A^G} A \to \operatorname{Map}(G, A)$

$$\sum_{i=1,\dots,n} a_i \otimes b_i \mapsto (g \mapsto \sum_{i=1,\dots,n} a_i(gb_i)), \ \ (a_i, b_i \in B,\ \forall g \in G) \tag{12}$$

From bijection of (10) it follows that bijectivity of can is equivalent to bijectivity of can^G, i.e

$$A \otimes_{A^G} A \approx \operatorname{Map}(G, A). \tag{13}$$

Lemma 4.4. *Let A be an unital algebra. Suppose that finite group G acts on A. Then following statements:*

1. $\operatorname{can}_G : A \otimes_{A^G} A \to \operatorname{Map}(G, A)$ *defined by (12) is bijection;*
2. *There are elements $b_i, a_i \in A$ ($i = 1, ..., n$) such that*

$$\sum_{i=1,\dots,n} a_i b_i = 1_A, \tag{14}$$

$$\sum_{i=1,\dots,n} a_i(gb_i) = 0\ \forall g \in G\ (g \text{ is nontrivial}); \tag{15}$$

are equivalent.

Proof. 1. => Denote by $e \in G$ unity of G. Let $f \in \operatorname{Map}(G, A)$ be such that

$$f(e) = 1_A;$$

$$f(g) = 0;\ (g \neq e).$$

From bijection of $A \otimes_{A^G} A \to \operatorname{Map}(G, A)$ it follows that there are elements $a_1, ..., a_n, b_1, ..., b_n \in A$ such that $\sum_{i=1,\dots,n} a_i \otimes b_i$ corresponds to f i.e.

$$f(g) = \sum_{i=1,\dots,n} a_i(gb_i).$$

It is clear that elements $a_1, ..., a_n, b_1, ..., b_n$ satisfy conditions (14), (15)

2. <= Let us enumerate elements of G, i.e $G = \{g_1, ..., g_{|G|}\}$. $a_1, ..., a_n, b_1, ..., b_n$ satisfy conditions (14), (15), and let be $f \in \operatorname{Map}(G, A)$ be any map from G to A; and $x \in A \otimes_{A^G} A$ is defined as

$$x = \sum_{i=1,\dots,|G|} f(g)a_i \otimes g^{-1}b_i.$$

From (14), (15) it follows that $f = \operatorname{can}_G(x)$ So can_G is map onto.

□

Definition 4.5. Let G be a finite group. Suppose that $H = \mathrm{Map}(G, \mathbb{C})$ and H has a natural structure of Hopf algebra. Any H - Galois extension $A \to B$ is said to be G - *Galois extension*.

4.6. Let G be a finite group $A^G \to A$ is a Hopf - Galois extension of C^* - algebras and A is unital. Let $\mathrm{Rep}(A)$ be a set of representations of A. The group G acts on $\mathrm{Rep}(A)$ by following way:

$$(g\rho)(a) = \rho(ga),\ a \in A,\ \rho \in \mathrm{Rep}(A),\ g \in G. \tag{16}$$

Lemma 4.7. *Action defined by (16) is free.*

Proof. Suppose that the action is not free. Then there are $\rho \in \mathrm{Rep}(A)$, $g \in G$ such that $g\rho = \rho$, i. e.

$$\rho(ga) = \rho(a),\ \forall a \in A. \tag{17}$$

Let $a_1, ..., a_n, b_1, ..., b_n \in A$ be elements which satisfies conditions (14) and (15). Then from (14) and (17) it follows that

$$\rho(\sum a_i g b_i) = \rho(\sum a_i b_i) = 1.$$

Otherwise from (15) it follows that

$$\rho(\sum a_i g b_i) = 0.$$

We have a contradiction. So action is free.

□

Corollary 4.8. *Let G be a finite group $A^G \to A$ is a Hopf - Galois extension and A is an unital commutative C^* - algebra. Then there is a covering projection of Hausdorff topological spaces.*

$$\Omega(A) \to \Omega(A^G).$$

Proof. Natural action of G on $\mathrm{Rep}(A)$ induces a natural action of G on a set $\mathrm{Irr}(A)$ of irreducible representations of A. However $\mathrm{Irr}(A) = \Omega(A)$. So G acts freely on $\Omega(A)$. □

Remark 4.9. Action of G on $\mathrm{Rep}(A)$ induces action of G on $\hat{A}$. However action of G should not be always free.

4.10. Let A be a C^* -algebra. From it follows that there is natural equivalence

4.3 Noncommutative finite covering projection

Definition 4.11. [18] Let A be C^* - algebra. A *- automorphism α is said to be *generalized inner* if is obtained by conjugating with unitaries from multiplier algebra $M(A)$.

Definition 4.12. [18] Let A be C^* - algebra. A *- automorphism α is said to be *partly inner* if its restriction to some non-zero α- invariant two-sided ideal is generalized inner. We call automorphism *purely outer* if it is not partly inner.

Definition 4.13. Suppose that finite group G acts on $\mathbb{M}_n(A)$. A projector $p \in \mathbb{M}_n(A)$ is said to be an G - *equivariant projector* if $gp = p$ for all $g \in G$. A left $A - G$ module P is said to be an *equivariant* G - *projective* if P is an image if G - equivariant projector.

Remark 4.14. Any connected commutative C^* - algebra corresponds to a connected topological space.

4.15. If $\rho : A \to B(H)$ a irreducible representation then denote by $\rho_\alpha : A \to B(H)$ a representation defined by following way

$$\rho_\alpha(a) = \rho(\alpha(a)).$$

Definition 4.16. Let A be C^* - algebra, α is *- automorphism of A. A representation $\pi : A \to B(H)$ is said to be α *-invariant representation* if ρ_α is unitary equivalent to ρ.

Definition 4.17. Let A be C^* - algebra, $\alpha \in \mathrm{Aut}(A)$, $\pi : A \to B(H)$ is α - invariant representation. We say that π is *inner with respect to* α if π is α - invariant representation and there exists a unitary element $u \in U(M(A))$ such that $\pi(\alpha(a)) = \pi(u)\pi(a)\pi(u^*)$ ($\forall a \in A$). If α is not inner for all α - invariant representations then α is said to be *strictly outer* automorphism.

Definition 4.18. Let A be a C^* - algebra and $G \subset \mathrm{Aut}(A)$ is a finite subgroup of * - automorphisms. An injective * - homomorphism $f : A^G \to A$ is said to be a *noncommutative finite covering projection* (or *noncommutative* G - *covering projection*) if f satisfies following conditions:

1. A is a finitely generated equivariant G - projective left and right A^G Hilbert C^*-module.
2. If $\alpha \in G$ then α is strictly outer
3. f is a G - Galois extension.
4. For any $\alpha \in \mathrm{Aut}(A^G)$ there is an automorphism $\beta \in \mathrm{Aut}(A)$ such that $\beta(a) = \alpha(a)$, $\forall a \in A^G$, i.e. $\beta|_{A^G} = \alpha$.

G is said to be *covering transformation group* of f. Denote by $G(B|A)$ covering transformation group of covering projection $A \to B$.

Definition 4.19. Let $A \to \mathcal{B}$ be a noncommutative finite covering projection such that:

1. algebra B is a finite direct sum of subalgebras, i.e. $B = \bigoplus_i A_i$;
2. There are injective * - homomorphisms $f_i : A \to B_i$ such that $f_i(a)b = ab$, ($a \in A$, $b \in B_i$);
3. Any * - homomorphisms $f_i : A \to B_i$ is a noncommutative finite covering projection

then $A \to B$ is said to be a *disconnected* noncommutative finite covering projection. A noncommutative finite covering projection is said to be *connected* is it is not disconnected.

Remark 4.20. If G is any finite group then for any C^* - algebra A one can construct disconnected G - covering projection. So disconnected covering projections do not supply any essential invariants. Thus disconnected covering projections are not interesing.

4.21. Let $f : A^G \to A$ be a noncommutative G - covering projection. Let $\rho : A \to \mathcal{B}(H)$ be a irreducible representation i. e. $\rho \in \hat{A}$. Let $g \in G$ and $\rho_g : A \to \mathcal{B}(H)$ is such that

$$\rho_g(a) = \rho(ga).$$

So it is an action of G on $\hat{A}$ such that

$$g \mapsto (\rho \mapsto \rho_g); \ \forall g \in G, \forall \rho \in \hat{A}. \tag{18}$$

Let us enumerate elements of G by integers, i. e. $g_1, ..., g_n \in G, \ n = |G|$ and define action of $\sigma : G \times \{i, ..., n\} \to \{i, ..., n\}$ such that $\sigma(g, i) = j \Leftrightarrow g_j = gg_i$ Let $\rho_\oplus = \oplus_{g \in G} \rho_g : A \to \mathcal{B}(H^n)$ be such that

$$\rho_\oplus(a)(h_1, ..., h_n) = (\rho(g_1 a)h_1, ..., (\rho(g_n a)h_n). \tag{19}$$

Let us define such linear action of G on H^n that

$$g(h_1, ..., h_n) = (h_{\sigma(g^{-1},1)}, ..., h_{\sigma(g^{-1},n)}). \tag{20}$$

From (19), (20) it follows that

$$g(ah) = (ga)(gh); \ \forall a \in A, \ \forall g \in G, \ \forall h \in H^n,$$

i.e. $H^n \in_A \mathcal{M}^G$. Representation $\rho_\oplus$ defines representation $\eta : A^G \to \mathcal{B}(K)$. $K = (H^n)^G$. If (η, H) is not irreducible then there is a nontrivial B - submodule $N \subsetneq K$. From ${}_A\mathcal{M}^G \approx_{A^G} \mathcal{M}$ it follows that $A \otimes_{A^G} N \subsetneq H^n$ is a nontrivial A - submodule. If we identify H with first summand of H^n then $(A \otimes_{A^G} K) \cap H \subsetneq H$ is a nontrivial A - submodule. This fact contradicts with that (ρ, H) is irreducible. So (η, K) is an irreducible representation. In result there is a natural map

$$\hat{f} : \hat{A} \to \hat{A}^G, \ (\rho \mapsto \eta) \tag{21}$$

and

$$\hat{A}^G \approx \hat{A}/G. \tag{22}$$

4.4 Covering projection of continuous trace C^*-algebras

Definition 4.22. [47] A positive element in C^* - algebra A is *abelian* if subalgebra $xAx \subset A$ is commutative.

Proposition 4.23. *[47] A positive element x in C^* - algebra A is abelian if* $\dim \pi(x) \leq 1$ *for every irreducible representation $\pi : A \to \mathcal{B}(H)$ of A.*

4.24. Let A be a C^* - algebra. For each $x \in A_+$ the (canonical) trace of $\pi(x)$ depends only on the equivalence class of an irreducible representation (π, H) of A, so that we may define a function $\hat{x} : \hat{A} \to [0, \infty]$ by $\hat{x}(t) = \text{Tr}(\pi(x))$ whenever $(\pi, H) \in t$. From Proposition 4.4.9 [47] it follows that $\hat{x}$ is lower semicontinuous function on a in Jacobson topology.

Definition 4.25. [47] We say that element $x \in A_+$ has *continuous trace* if $\hat{x} \in C^b(\hat{A})$. We say that C^* - algebra has *continuous trace* if set of elements with continuous trace is dense in A_+.
We say that a C^* - algebra A is of type I if each non-zero quotient of A contains non-zero abelian element. If A is even generated (as C^* - algebra) by its Abelian elements we say that it is of type I_0.

Lemma 4.26. *Let $\phi : A \to B$ be an injective *- homomorphism of C^* -*

Lemma 4.27. *If A is a C^* - algebra, $G \subset \text{Aut}(A)$ is a finite group, A^G is continuous trace algebra then A is a continuous trace algebra.*

Theorem 4.28. *(Theorem 5.6 [47]) For each C^* - algebra A there is a dense hereditary ideal $K(A)$, which is minimal among dense ideals.*

Proposition 4.29. *[47] Let A be a C^* - algebra with continuous trace Then*

1. *A is of type I_0;*
2. *$\hat{A}$ is a locally compact Hausdorff space;*
3. *For each $t \in \hat{A}$ there is an abelian element $x \in A$ such that $\hat{x} \in K(\hat{A})$ and $\hat{x}(t) = 1$.*

The last condition is sufficient for A to have continuous trace.

Remark 4.30. From [13], Proposition 10, II.9 it follows that a continuous trace C^*-algebra is always a CCR-algebra, a C^*-algebra where for every irreducible representation (π, H) of A and for every element $x \in A$, $\pi(x)$ is a compact operator.

Lemma 4.31. *[47] Let G be a finite group and $f : A^G \to A$ is a G - covering projection. If A^G is a continuous trace C^* - algebra then A is also a continuous trace C^* - algebra*

Proof. From 4.21 it follows that for any irreducible representation $\rho : A \to \mathcal{B}(H)$ there is a irreducible representation $\eta : A^G \to \mathcal{B}(H)$ such that

$$\rho|_{A^G} = \eta \tag{23}$$

Let $x \in A^G$ be an abelian element of A^G. From 4.23 it follows that $\dim \eta(x) \leq 1$ for any irreducible representation $\eta : A^G \to \mathcal{B}(H)$. From (23) it follows that $\dim \rho(x) \leq 1$ for any irreducible representation $\rho : A \to \mathcal{B}(H)$. So any abelian element of A^G is an abelian element of A. Let $t \in \hat{A}$ be an element of spectrum and $s = \hat{f}(t) \in \hat{A}^G$ where $\hat{f}$ is defined by (21). From 4.23 it follows that there is an abelian element $x \in A^G$ such that $\hat{x} \in K(\hat{A}^G)$ and $\hat{x}(s) = 1$. However x is a abelian element of A, $\hat{x} \in K(A)$ and $\hat{x}(t) = \hat{x}(s) = 1$. From 4.29 it follows that A is a continuous trace C^* - algebra.

□

Proposition 4.32. *[11] If a topological group G act properly on a topological space then orbit space X/G is Hausdorff. If also G is Hausdorff, then X is Hausdorff.*

Lemma 4.33. *Let G be a finite group, $f : A^G \to A$ is a noncommutative G- covering projection and A^G is separable continuous trace algebra. Then G acts freely on $\hat{A}$, there is a natural homeomorphism $\hat{A}^G \approx \hat{A}/G$ which is a topological covering projection.*

Proof. Suppose that G does not act freely on $\hat{A}$. Then there are $x \in \hat{A}$ and $g \in G$ such that $gt = t$ $(t \in \hat{A})$. By definition 4.18 g should be strictly outer. Let $\rho : A \to \mathcal{B}(H)$ be representative of x. Then (ρ_g, H) is also representative of x. So ρ is unitary equivalent to ρ_g, i. e. there is unitary $U \in U(H)$ such that $\rho_g(a) = U\rho(a)U^*$ $(\forall a \in A)$. From 4.31 it follows that A is continuous trace algebra. So $\rho(A) = \mathcal{K}(H)$, $\rho(M(A)) = B(H)$, $\rho(U(M(A))) = U(H)$. So it is $u \in M(A)$ such that $\rho(u) = U$ and we have $\rho_g(a) = \rho(u)\rho(a)\rho(u^*)$. It means that g is inner with respect to ρ, so action of g is not strictly outer. This contradiction proves the lemma. □

Remark 4.34. From lemma 4.33 it follows that finite covering projections of commutative algebras are just covering projections of their character spaces. If A^G is a commutative C^* - algebra then dim $\pi(A^G) = 1$ for all irreducible $\pi : A \to \mathcal{B}(H)$. If $f : A^G \to A$ is noncommutative G covering projection and A^G is commutative then A^G is continuous trace algebra $\Omega(A^G) \approx \hat{A}^G$. From 4.31 it follows that A is also a continuous trace C^* - algebra. If $\rho : A \to \mathcal{B}(H)$ then $\rho(A) = \mathcal{K}(H)$. Let us recall construction from 4.21. Let us enumerate elements of G by integers, i. e. $g_1, ..., g_n \in G$, $n = |G|$ and define action of $\sigma : G \times \{i, ..., n\} \to \{i, ..., n\}$ such that $\sigma(g, i) = j \Leftrightarrow g_j = gg_i$ Let $\rho_\oplus = \oplus_{g \in G}\rho_g : A \to \mathcal{B}(H^n)$ be such that

$$\rho_\oplus(a)(h_1, ..., h_n) = (\rho(g_1 a)h_1, ..., (\rho(g_n a)h_n). \tag{24}$$

Let us define such linear action of G on H^n that

$$g(h_1, ..., h_n) = (h_{\sigma(g^{-1},1)}, ..., h_{\sigma(g^{-1},n)}). \tag{25}$$

From (19), (25) it follows that

$$g(ah) = (ga)(gh); \ \forall a \in A, \ \forall g \in G, \ \forall h \in H^n,$$

i.e. $H^n \in_A \mathcal{M}^G$. Representation $\rho_\oplus$ defines representation $\eta : A^G \to \mathcal{B}(K)$. $K = (H^n)^G$. From 4.21 η is irreducible representation and since A^G is commutative it follows that dim $K = 1$. From (25) it follows that dim $H = 1$. Thus the dimension of any irreducible representation of A equals to 1. It means that any irreducible representation is commutative. From this fact it follows that A is a commutative C^* - algebra $\hat{A} = \Omega(A)$ and $\Omega(f) : \Omega(A) \to \Omega(A^G)$ is a (topological) covering projection.

4.5 Covering projections of noncommutative torus

Let us consider Galois extensions of noncommutative torus. Noncommutative torus A_θ is C^* - norm completion of algebra generated by two unitary elements u, v and following conditions are hold:

$$uu^* = u^*u = vv^* = v^*v = 1;$$

$$uv = e^{2\pi i\theta} vu,$$

where $\theta \in \mathbb{R}$. If $\theta = 0$ then $A_\theta = A_0$ is commutative algebra of continuous functions on commutative torus $C(S^1 \times S^1)$ There is such trace τ_0 on A_θ that $\tau_0(\sum_{-\infty<i<\infty, -\infty<j<\infty} a_{ij} u^i v^j) = a_{00}$. C^* - norm of A_θ is defined by following way $\|a\| = \sqrt{\tau_0(a^* a)}$. Let us consider * - homomorphism $f : A_\theta \to A_{\theta'}$, where $A_{\theta'}$ is generated by unitary elements u' and v'. Homomorphism f is defined by following way:

$$u \mapsto u'^m;$$

$$v \mapsto v'^n;$$

It is clear that

$$\theta' = \frac{\theta + k}{mn};\ (k = 0, ..., mn - 1). \tag{26}$$

Let us show that *-homomorphism f is Galois extension. First of all note that commutative $C*$- subalgebras $C(u') \subset A_{\theta'}$ and $C(v') \subset A_{\theta'}$ generated by u' and v' respectively are isomorphic to algebra $C(S^1)$, where S^1 is one dimensional circle. There are induced by f *-homomorphisms $C(S^1) = C(u) \to C(u') = C(S^1)$, $C(S^1) = C(v) \to C(v') = C(S^1)$. These *-homomorphisms induces m and n listed covering projections respectively. Covering groups of these covering projections are $G_1 \approx \mathbb{Z}_m$ and $G_2 \approx \mathbb{Z}_n$ respectively. Generators of these groups are presented below:

$$u' \mapsto e^{\frac{2\pi i}{m}} u';$$

$$v' \mapsto e^{\frac{2\pi i}{n}} v'.$$

Homomorphisms of commutative algebras $C(u) \to C(u')$, $C(v) \to C(v')$ correspond to covering projection, it follows that there are elements $x_i \in C(u')$ $(i = 1, ..., r)$, $y_j \in C(v')$ $(j = 1, ..., s)$ such that

$$\sum_{1 \le i \le r} x_i^2 = 1_{C(u')};$$

$$\sum_{1 \le i \le r} (g_1 x_i) x_i = 0; g_1 \in G_1;$$

$$\sum_{1 \le j \le s} y_i^2 = 1_{C(v')};$$

$$\sum_{1 \le j \le s} (g_2 y_i) y_i = 0; g_2 \in G_2,$$

where g_1 and g_2 are nontrivial elements of G_1 and G_2.
Actions of G_1 and G_2 induce action of $G = G_1 \times G_2$ on $A_{\theta'}$. Let us set

$$a_k = y_j x_i,$$

$$b_k = x_i y_j,$$

where $k = im + j$.

It is easy to check following equalities.

$$\sum_{1\le k\le mn} a_k b_k = 1_{A_{\theta'}};$$

$$\sum_{1\le k\le mn} (ga_k) b_k = 0,$$

where $g \in G$ is nontrivial element. We would like to prove this example covers whole set of noncommutative torus covering projections. Let A_θ be a noncommutative torus, generated by unitaries $u, v \in U(A_\theta)$ The commutative torus $\mathbb{T}^2 = U(1) \times U(1)$ exactly acts on A_θ by following way:

$$(z_1, z_2)u = z_1 u,\ (z_1, z_2)v = z_2 v,\ (z_1, z_2) \in \mathbb{T}^2 = U(1) \times U(1).$$

Let $A_\theta \to B$ be G - Galois extension, A_θ is considered as subalgebra of B, i. e. $A_\theta \subset B$. Let $G' = \{g \in \mathrm{Aut}(B) : g|_{A_\theta} \in \mathbb{T}^2\}$. There is following exact sequence of groups:

$$\{e\} \to G \to G' \xrightarrow{f} \mathbb{T}^2 \to \{e\}. \tag{27}$$

Homomorphism f is a covering projection (in topological sense) because G is a finite group. Let us consider following special cases of sequence (27):

1. $G' = G \times \mathbb{T}^2$
2. G' is a connected topological space.

4.35. $G' = G \times \mathbb{T}^2$
In this case we have following:

$$G' \approx \bigoplus_{g\in G} \mathbb{T}^2_g,$$

$$f((t_{g_1}, ..., t_{g_n})) = t_{g_1} + ... + t_{g_n},\ (t_{g_1}, ..., t_{g_n}) \in \bigoplus_{g\in G} \mathbb{T}^2_g.$$

The $\bigoplus_{g\in G} \mathbb{T}^2_g$ is a compact Lie group and any its representation is a direct sum of irreducible representations. Any irreducible representation $\bigoplus_{g\in G} \mathbb{T}^2_g \to U(1)$ is given by:

$$((z_{1g_1}, z_{2g_1}), ..., (z_{1g_n}, z_{2g_n})) \mapsto z_{1g_1}^{i_{g_1}} z_{2g_1}^{j_{g_1}} \dots z_{1g_n}^{i_{g_n}} z_{2g_n}^{j_{g_n}},$$

$$z_{1g_k}, z_{2g_k} \in U(1),\ (z_{1g_k}, z_{2g_k}) \in \mathbb{T}_{g_k},\ i_{g_k}, j_{g_k} \in \mathbb{Z}.$$

An element $a \in B$ is said to be a *homogeneous* element of type $((i_{g_1}, j_{g_1}), ..., (i_{g_n}, j_{g_n}))$ if it satisfies following condition:

$$((z_{1g_1}, z_{2g_1}), ..., (z_{1g_n}, z_{2g_n}))a = z_{1g_1}^{i_{g_1}} z_{2g_1}^{j_{g_1}} \dots z_{1g_n}^{i_{g_n}} z_{2g_n}^{j_{g_n}} a.$$

is said to be of type $((i_{g_1}, j_{g_1}), ..., (i_{g_n}, j_{g_n}))$ $(i_{g_k}, j_{g_k})))$. If a' (resp. a'') is a homogeneous element of type $((i'_{g_1}, j'_{g_1}), ..., (i'_{g_n}, j'_{g_n}))$, (resp. $((i''_{g_1}, j''_{g_1}), ..., (i''_{g_n}, j''_{g_n})))$ then the product $a'a''$

is a homogeneous element of type $((i'_{g_1}+i''_{g_1}, j'_{g_1}+j''_{g_1}), ..., (i'_{g_n}+i''_{g_n}, j'_{g_n}+j''_{g_n}))$. So B is a $\left(\mathbb{Z}^2\right)^G$ graded algebra. G naturally acts on $\left(\mathbb{Z}^2\right)^G$. If $x \in \left(\mathbb{Z}^2\right)^G$ and $a \in B$ is homogeneous element of type x then ga is a homogeneous element of type gx. Similarly A_θ is a $\mathbb{Z}^2$ graded alebra and we for all $x \in \mathbb{Z}^2$ we can define x homogeneous elements. From exactness of $\bigoplus_{g\in G}\mathbb{T}^2_g$ action it follows that there is a nonzero homogeneous element $u_{g1} \in B$ of type $((1,0),(0,0),...,(0,0)$. Denote by u_g a homogeneous element given by:

$$u_g = g' u_{g_1},\ g' g_1 = g \in G.$$

There is the $\mathbb{C}$ - linear map $p : B \to A_\theta$ given by:

$$p(a) = \sum_{g\in G} ga, \forall a \in A_0.$$

It is clear that $p(u_{g1}) \in A_\theta$ is a $(0,1)$ homogeneous element. However any $(0,1)$ homogeneous element is equal to cu $(c \in \mathbb{C})$. If we replace u_{g_1} with $c^{-1}u_{g_1}$ then $p(u_{g_1}) = u$. From $p(u_{g1})p(u^*_{g1}) = uu^* = 1$ it follows that

$$(u_{g_1} + ... + u_{g_n})(u^*_{g_1} + ... + u^*_{g_n}) = 1. \tag{28}$$

Right part of (28) is a $((0,0),...,(0,0))$ homogeneous element. If $u_{g_1}u^*_{g_2} \neq 0$ then left part of (28) contains a nonzero homogeneous summand of $((1,0),(0,-1),...,(0,0))$ type. Because it is impossible we have $u_{g_1}u^*_{g_2} = 0$. Similarly we can define elements v_{g_1}, v_{g_n} and

$$v_{g1} + ... + v_{g_n} = v;$$

$$(v_{g1} + ... + v_{g_n})(v^*_{g1} + ... + v^*_{g_n}) = 1.$$

If $u_{g_1}v_{g_2} \neq 0$ then right part of

$$uv = (u_{g1} + ... + u_{g_n})(v_{g1} + ... + v_{g_n}) \tag{29}$$

contains a nonzero homogeneous summand of $((1,0),(0,1),(0,0),...,(0,0))$ type. However right part of (29) cannot contain this summand, so $u_{g_1}v_{g_2} = 0$. Similarly if $g', g'' \in G$ and $g' \neq g''$ we have following:

$$u_{g'}u_{g''} = u_{g'}u^*_{g''} = u^*_{g'}u_{g''} = u^*_{g'}u^*_{g''} = v_{g'}v_{g''} = v_{g'}v^*_{g''} = v^*_{g'}v_{g''} = v^*_{g'}v^*_{g''} = 0, \tag{30}$$

$$u_{g'}v_{g''} = u_{g'}v^*_{g''} = u^*_{g'}v_{g''} = u^*_{g'}v^*_{g''} = v_{g'}u_{g''} = v_{g'}u^*_{g''} = v^*_{g'}u_{g''} = v^*_{g'}u^*_{g''} = 0. \tag{31}$$

Element $uu_{g_1}u^*_{g_1}$ is a sum of homogeneous elements of $((1,0),(0,0),...,(0,0))$, ..., $((0,0),(1,0),...,(0,0))$,..., $((0,0),(0,0),...,(1,0))$ However from (30), (31) it follows that summands of $((0,0),(1,0),...,(0,0))$,..., $((0,0),(0,0),...,(1,0))$ types vanish, so we have:

$$uu_{g_1}u^*_{g_1} = u_{g_1}$$

or

$$ue_{g_1} = u_{g_1}$$

where $e_{g_1} = u_{g_1} u^*_{g_1}$. Similarly we can introduce e_g for all $g \in G$. From previous equations it follows that e_g is an idempotent and B is a following direct sum of algebras:

$$B = \bigoplus_{g \in G} e_g B,$$

i.e. B is a directed sum of algebras. It means that $A_\theta \to B$ is a disconnected noncommutative covering projection.

4.36. G' is a connected topological space.
In this case G' is a commutative torus $\mathbb{T}^2 = U(1) \times U(1)$. Homomorphism $f : G' \to \mathbb{T}^2$ is given by:

$$(z_1, z_2) \to (z_1^n, z_2^m)$$

where $(z_1, z_2) \in U(1) \times U(1) \approx G'$, $(n, m \in \mathbb{N})$. Elemets of G in G' are given by:

$$\left(e^{\frac{2\pi i k_1}{n}}, e^{\frac{2\pi i k_2}{m}}\right), \ (k_1, k_2 \in \mathbb{Z}).$$

Element $a \in B$ is said to be a homogeneous of degree (r, s) if it satisfies following condition:

$$(z_1, z_2)a = z_1^r, z_1^s.$$

From exactness of G' action it follows that there exist a homogeneous of degree $(1, 0)$ nonzero element $u' \in B$. Element u'^n is G invariant and homogeneous of degree $(n, 0)$. It is clear that $u'^n = cu$ $(c \in \mathbb{C})$. Similarly we can prove that there is an element v' such that $v'^m = v$. In this case we have finite noncommutative covering projection from example 4.41.

4.6 Invariants of finite covering projections

Let $\mathcal{C}$ be a category and $F_!, F^! : \mathcal{C} \to \mathfrak{Groups}$ covariant and contravariant functors from $\mathcal{C}$ to category to category of groups such that it is natural isomorphism $F_!(A) \approx F^!(A)$ for any object in $\mathcal{C}$. If f is morphism in $\mathcal{C}$ then $F^! \circ F_!(f)$ is endomorphism of $F_!(A)$

Definition 4.37. Endomorphism $F^! \circ F_!(f)$ is said to be *invariant of f with respect to pair* $(F_!, F^!)$.

Example 4.38. There is a *standard odd grading* on $A \otimes A$ for any algebra A: $(A \oplus A)^{(0)} = \{(a, a) : a \in A\}$ and $(A \oplus A)^{(1)} = \{(a, -a) : a \in A\}$ So any C^* algebra may by regarded as $\mathbb{Z}_2$ - graded algebra. If A and B are $\mathbb{Z}_2$ - graded a C^* algebras. Let $\mathbb{E}(A, B)$ is set of triples of triples (E, ϕ, F) such that E is countably generated graded Hilbert module over B, ϕ is a graded *- homomorphism from A to $\mathcal{B}(E)$, and F is an operator in $\mathcal{B}(E)$ of degree 1, such that $[F, \phi(a)]$, $(F^2 - 1)\phi(a)$ and $(F - F^*)\phi(a)$ are all in $\mathcal{K}(E)$ for all $a \in A$. Triple (E, ϕ, F) is called a $KK(A, B)$ cycle. There is equivalence relation $\approx$ on $\mathbb{E}(A, B)$ defined in [28] $KK(A, B) = \mathbb{E}(A, B)/\approx$. Elements of $KK(A, B)$ are equivalence classes of triples (E, ϕ, F) such that E is countably generated graded Hilbert module over B, ϕ is a graded

*- homomorphism from A to $\mathcal{B}(E)$, and F is an operator in $\mathcal{B}(E)$ of degree 1, such that $[F, \phi(a)]$, $(F^2-1)\phi(a)$ and $(F-F^*)\phi(a)$ are all in $\mathcal{K}(E)$ for all $a \in A$. Triple (E, ϕ, F) is called a $KK(A, B)$ cycle. If $f : A_1 \to A_2$ is graded homomorphism then for any B there is group homomorphism $f^* : KK(A_2, B) \to KK(A_1, B)$ induced by map $(E, \phi, F) \to (E, \phi \circ f, F)$. If $f : A_1 \to A_2$ is injective graded homomorphism such that A_2 is finitely generated A_1 module, (E, ϕ, F) is $KK(A_1, B)$ cycle then $(A_2 \otimes_f E, 1 \otimes \phi, 1 \otimes F)$ is $KK(A_2, B)$ cycle. So there is wrong way functoriality homomorphism $f_! : KK(A_1, B) \to KK(A_2, B)$. It is clear that $f^* f_! \in \mathrm{End}_{KK(A_1, A_1)}(KK(A_1, B))$. If $g : B_1 \to B_2$ is graded homomorphism then for any A there is group homomorphism $g_* : KK(A, B_1) \to KK(A, B_2)$ induced by map $(E, \phi, F) \mapsto (E \hat{\otimes}_g B_2, \phi \hat{\otimes} 1, F \hat{\otimes} 1)$, where $\hat{\otimes}$ means Hilbert tensor product. If B_2 is finitely generated B_1 module then any $KK(B_2, A)$ cycle (E, ϕ, F) can be regarded as $KK(B_1, A)$ cycle $(E, \phi \circ g, F)$. So there is wrong way functorality homomorphism $g^! : KK(A, B_2) \to KK(A, B_1)$. It is clear that $g^! g_* \in \mathrm{End}_{KK(B_1, B_1)}(KK(A, B_1))$.

Definition 4.39. Let $f : B \to A$ be a noncommutative finite covering and C is separable C^* - algebra. Endomorphism $f^* f_! \in \mathrm{End}_{KK(B,B)}(KK(B, C))$ (resp $f^! f_*) \in \mathrm{End}_{KK(B,B)}(KK(C, B))$ is said to be *contravariant* (resp. *covariant*) *invariant of* f. By $[f, C]$ and $[C, f]$ denote these invariants.

Example 4.40. Let A be algebras of continuous trace, $Q^s(A) = M(A \otimes \mathcal{K}) / A \otimes \mathcal{K}$ is stable multiplier algebra [28] of A. There is natural injective homomorphism $C_0(\hat{A}) \to Q^s(A)$. So any $KK(Q^s(A), B)$ cycle may be regarded as $KK(C_0(\hat{A}), B)$ cycle. If $\mathbb{C}_1 \approx \mathbb{C}^2$ algebra with odd grading then there is canonical homomorphism $KK * (C_0(\hat{A}), \mathbb{C}_1)$. Otherwise since $KK^*(S^1, \mathbb{C}_1) \approx \mathbb{Z}$ we have canonical homomorphism $\pi_1(C_0(\hat{A})) \to KK^*(C_0(\hat{A}), \mathbb{C}_1)$. Thus it is canonical homomorphism $\phi : \pi_1(C_0(\hat{A})) \to KK(Q^s(A), \mathbb{C}_1)$. So ϕ is natural transformation from $(\pi_1, p_1^!)$ to $(F_!, F^!)$.

Example 4.41. Let $\theta \in \mathbb{R}$ be irrational number, $m, n \in \mathbb{N}$, $mn > 1$, $\theta' = \theta / mn$, $\theta'' = (\theta + k)/mn$ ($k \neq 0 \bmod mn$). Let $u, v \in A_\theta$, $u', v' \in A_{\theta'}$, $u'', v'' \in A_{\theta''}$ be generators of noncommutative toruses. Let $f' : A_\theta \to A_{\theta'}$ (resp. $f'' : A_\theta \to A_{\theta''}$) be * - homomorphism $u \mapsto u'^m$, $v \mapsto v'^n$ (resp. $u \mapsto u''^m$, $v \mapsto v''^n$). It is known that $K^1(A_\theta) \approx KK(C_0(R), A_\theta) \approx \mathbb{Z}^2$, and $[u], [v] \in K^1(A_\theta$ are generators of $K^1(A_\theta)$. Covariant invariant $[C_0(\mathbb{R}), f']$ satisfies following equation

$$[u] \mapsto m[u], [v] \mapsto n[v].$$

It is clear that $[C_0(\mathbb{R}), f'] = [C_0(\mathbb{R}), f'']$. Covering transformation groups of both f' are f'' are isomorphic to $\mathbb{Z}_m \times \mathbb{Z}_n$. This example shows noncommutative torus has much more coverings than commutative ones.

5 Towards noncommutative homotopy theory

According to example 4.41 there are a lot of noncommutative covering projections besides commutative ones. This fact can substantially extend calculated by coverings fundamental group. However this group can be reduced if set of finite coverings is factorized by equivalence relation. This relation is Morita equivalences induced by bimodules over Hopf-

Galois extensions. Defined in 4.39 invariants are invariant with respect to this equivalence relation.

5.1 Morita equivalences induced by bimodules over Hopf-Galois extensions

5.1.1 General theory

Definition 5.1. Let A and B be algebras. A *Morita context* connecting A and B is a sextuple $\mathcal{M} = (A, B, M, N, \alpha, \beta)$

1. $M \in {}_A\mathcal{M}_B$,
2. $N \in {}_B\mathcal{M}_A$,
3. $\alpha : M \otimes_B N \to A$ (α is a morphism in ${}_A\mathcal{M}_A$) ,
4. $\beta : N \otimes_A M \to B$ (β is a morphism in ${}_B\mathcal{M}_B$) .

such that

1. $\alpha(x \otimes y)x' = x\beta(y \otimes x')$;
2. $\beta(y \otimes x)y' = y\alpha(a \otimes y')$; $\forall x, x' \in M, \ \forall y, y' \in N$.

Definition 5.2. A Morita context $\mathcal{M} = (A, B, M, N, \alpha, \beta)$ is *strict* if both α and β are isomorphisms.

Let M be a right H-comodule, and N a left H-comodule. The cotensor product $M\Box_H N$ is the $\mathbb{C}$ - module

$$M\Box_H N = \{\sum_i m_i \otimes n_i \in M \otimes N \mid \sum_i \rho(m_i) \otimes n_i = \sum_i m_i \otimes \lambda(n_i)\}. \tag{32}$$

If H is cocommutative, then $M\Box_H N$ is also a right (or left) H-comodule. If G is finite group and $H = C(G)$ then

$$M\Box_H N = M\Box_G N \subset M \otimes N,$$

where $M\Box_G N$ is generated by elements $a \otimes b \in M \otimes N$ such that

$$ag \otimes b = a \otimes gb; \ \forall g \in G.$$

Definition 5.3. [22] Let A and B be right H-comodule algebras. An *H-Morita context* connecting A and B is a Morita context $(A, B, M, N, \alpha, \beta)$ such that $M \in {}_A\mathcal{M}_B^H$, $N \in {}_B\mathcal{M}_A^H$, $\alpha : M \otimes_A N \to A$ is a morphism in ${}_A\mathcal{M}_B^H$ and $\beta : N \otimes_B M \to A$ is a morphism in ${}_B\mathcal{M}_A^H$

A morphism between two H-Morita contexts $(A, B, M, N, \alpha, \beta)$ and $(A', B', M', N', \alpha', \beta')$ is defined in the obvious way: it consists of a quadruple $(\kappa, \lambda, \mu, \nu)$, where $\kappa : A \to A'$ and $\lambda : B \to B'$ are H-comodule algebra maps, $\mu : M \to M'$ is a morphism in ${}_A\mathcal{M}^H_B$ and $\nu : N \to N'$ is a morphism in ${}_B\mathcal{M}^H_A$ such that $\kappa \circ \alpha = \alpha'(\mu \otimes \nu)$ and $\lambda \circ \beta = \beta'(\nu \otimes \mu)$ $\underline{\underline{\text{Morita}}}^H(A, B)$ will be the subcategory of the category of H-Morita contexts, consisting of H-Morita contexts connecting A and B, and morphisms with the identity of A and B as the underlying algebra maps.

Lemma 5.4. *Let $(A, B, M, N, \alpha, \beta)$ be a strict H-Morita context. Then we have a pair of inverse equivalences $(M \otimes_B -, N \otimes_A -)$ between the categories ${}_A\mathcal{M}^H$ and ${}_B\mathcal{M}^H$.*

Proof. Let $P \in_B \mathcal{M}^H$. Then $M \otimes_B P \in_B \mathcal{M}^H$, with right H-action

$$\rho(m \otimes_B p) = m_{[0]} \otimes_B p_{[0]} \otimes m_{[1]} p_{[1]}.$$

The rest of the proof is straightforward. □

Definition 5.5. [22] Assume that A and B are right faithfully flat H-Galois extensions of $A^{\mathrm{co}H}$ and $B^{\mathrm{co}H}$. A H-Morita context between $A^{\mathrm{co}H}$ and $B^{\mathrm{co}H}$ is a Morita context ($A^{\mathrm{co}H}$ and $B^{\mathrm{co}H}, M_1, N_1, \alpha_1, \beta_1)$ such that M_1 (resp. N_1) is a left $A\Box_H B^{\mathrm{op}}$-module (resp. $B\Box_H A^{\mathrm{op}}$-module) and

$$\alpha_1 : M_1 \otimes_{B^{\mathrm{co}H}} N_1 \to A^{\mathrm{co}H} \text{ is left } A\Box_H A^{\mathrm{op}} - \text{linear},$$

$$\beta_1 : N_1 \otimes_{A^{\mathrm{co}H}} M_1 \to B^{\mathrm{co}H} \text{ is left } B\Box_H B^{\mathrm{op}} - \text{linear}.$$

A morphism between two $\Box_H$-Morita contexts connecting $A^{\mathrm{co}H}$ and $A^{\mathrm{co}H}$ is a morphism between Morita contexts of the form $(A^{\mathrm{co}H}, B^{\mathrm{co}H}, \mu_1, \nu_1)$, where μ_1 is left $A\Box_H B^{\mathrm{op}}$-linear and ν_1 is left $B\Box_H A^{\mathrm{op}}$-linear. The category of $\Box_H$-Morita contexts connecting $A^{\mathrm{co}H}$ and $A^{\mathrm{co}H}$ will be denoted by $\underline{\underline{\text{Morita}}}^{\Box_H}(A^{\mathrm{co}H}, B^{\mathrm{co}H})$.

Theorem 5.6. *Let A and B be right faithfully flat H-Galois extensions of $A^{\mathrm{co}H}$ and $A^{\mathrm{co}H}$. Then the categories $\underline{\underline{\text{Morita}}}^H(A, B)$ and $\underline{\underline{\text{Morita}}}^{\Box_H}(A^{\mathrm{co}H}, B^{\mathrm{co}H})$ are equivalent. The equivalence functors send strict contexts to strict contexts.*

Definition 5.7. Let B and C be right H-comodule algebras such that $B^{\mathrm{co}H} = C^{\mathrm{co}H} = A$. So M and N are $A - A$ bimodules. Suppose that $(B, C, M, N, \alpha, \beta)$ is an H - Morita context (see definition 5.3). The $\mathcal{M}$ is called an *$A - H$-Morita context* if both $\alpha : M \otimes_C N \to B$ and $\beta : N \otimes_C M \to B$ are morphisms in ${}_A\mathcal{M}_A$, i. e. morphisms of $A - A$ bimodules.

If we concern with C^* algebras then notion of strong Morita equivalence [15],[39] is rather adequate then Morita equivalence. According to [39] Morita equivalence of C^* algebras A and B is equivalent to stable isomorphism, i.e.

$$A \otimes \mathcal{K} \approx B \otimes \mathcal{K}$$

5.1.2 Action of finite groups

Following definition is adoption of 5.7 to noncommutative finite coverings.

Definition 5.8. Covering projections $A \to B$, $A \to C$ are *Morita equivalent* or G - *Morita equivalent* if

1. Both covering projections have the same covering transformation group G.
2. There is G equivariant isomorphism $B \otimes \mathcal{K} \approx C \otimes \mathcal{K}$. This isomorphism is also $A - A$ bimodule isomorphism.

Remark 5.9. From $\mathcal{K} \otimes \mathcal{K} \approx \mathcal{K}$ it follows that for for all $f : A \to B \otimes \mathcal{K}$ it is *- homomorphism $f : A \otimes \mathcal{K} \to B \otimes \mathcal{K}$. Conversely if $p \in \mathcal{K}$ is one dimensional projection then $(\mathrm{Id}_A \otimes p) A \otimes \mathcal{K} (\mathrm{Id}_A \otimes p) \approx A$ so we have correspondence between *- homomorphisms $A \otimes \mathcal{K} \to B \otimes \mathcal{K}$ and $A \otimes \mathcal{K} \to B \otimes \mathcal{K}$. If $A \to B$, $A \to C$ are G - Morita equivalent covering projection then there is G - equivariant *- homomorphism $B \to C \otimes \mathcal{K}$ which is $A - A$ bimodule homomorphism.

Example 5.10. Consider finite coverings $A_\theta \to A_{\theta'}$, $A_\theta \to A_{\theta''}$ from 4.41, $\theta' = \theta/mn$, $\theta'' = (\theta + k)/mn$. Covering transformation group of both coverings is $G = \mathbb{Z}_m \otimes \mathbb{Z}_n$. Let $U, V \in \mathbb{M}_{N=mn}(\mathbb{C})$ be unitary matrixes such that

$$UV = e^{2\pi i k/nm} VU.$$

There is following G equivariant isomorphism $A_{\theta'} \otimes \mathbb{M}_N(\mathbb{C}) \approx A_{\theta''} \otimes \mathbb{M}_N(\mathbb{C})$

$$u' \otimes 1 \to u'' \otimes U;\ v' \otimes 1 \to v'' \otimes V.$$

This isomorphism is also $A_\theta - A_\theta$ bimodule isomorphism. From $\mathcal{K} \otimes \mathbb{M}_N(\mathbb{C}) \approx \mathcal{K}$ it follows that there exist isomorphism $A_{\theta'} \otimes \mathcal{K} \approx A_{\theta''} \otimes \mathcal{K}$ such that conditions of definition 5.8 are satisfied. So considered coverings are Morita equivalent.

5.2 Category of finite noncommutative covering projections

We would like construct category such that Morita equivalent covering projections are isomorphic. So we replace any C^* - algebra A by $A \otimes \mathcal{K}$ Objects of this category are finite noncommutative finite covering projections. Let A - be a stable C^* - algebra, $f_1 : A_1^{G_1} \to A_1$, $f_2 : A_2^{G_2} \to A_2$ are noncommutative covering projections such that $A_1^{G_1} \approx A_2^{G_2} \approx A$, $k : G_2 \to G_1$ is a surjective group homomorphism. Consider *- homomorphisms $f : A_1 \to A_2$ such that

1. $f(k(g)a) = gf(a);\ \forall g \in G_2,\ \forall a \in A_1$.
2. f is $A - A$ bimodule homomorphism.

Definition 5.11. Let $A \to A_1$, $A \to A_2$ be to finite noncommutative covering projection. An A *covering morphism* is a *- homomorphisms $f : A_1 \to A_2$ such that above conditions are satisfied.

5.12. Let A be a stable C^* algebra. Let us introduce category $\mathrm{Cov}(A)$ of A - covering projections. Objects of the category are noncommutative finite covering projections, morphisms are A - covering morphisms.

Definition 5.13. Above category is said to be *category of A - covering projections*.

5.14. Fundamental group functor[34] is a functor from category of topological sets to category of groups. This functor is defined by following way

$$X \mapsto \pi_1(X),$$
$$f : X \to Y \mapsto \pi_1(f) : \pi_1(X) \to \pi(Y).$$

Noncommutative generalization of fundamental group $\pi_1(X)$ is not known yet. However we know generalization of covering group. So one can construct generalization of $\pi_1(f)$ with respect to covering. First of all we define π_1 with respect to covering in commutative case.
Let $f : X \to Y$ be continuous map, and $\widetilde{X} \to X$, $\widetilde{Y} \to Y$ such normal coverings that following diagram

$$\begin{array}{ccc} \widetilde{X} & \longrightarrow & \widetilde{Y} \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

This diagram induces following diagram with surjective vertical arrows.

$$\begin{array}{ccc} \pi_1(X) & \longrightarrow & \pi_1(Y) \\ \downarrow & & \downarrow \\ G(\widetilde{X}|X) & \longrightarrow & G(\widetilde{Y}|Y) \end{array}$$

Definition 5.15. Let us consider above coverings. Homomorphism $G(\widetilde{X}|X) \to G(\widetilde{Y}|Y)$ is called *Fundamental group homomorphism* with respect to coverings $\widetilde{X} \to X$, $\widetilde{Y} \to Y$.

This definition have noncommutative generalization.

Definition 5.16. Let A, B be C^* algebras, $f : A \to B$ * - homomorphism, $A \to \widetilde{A}$, $B \to \widetilde{B}$ noncommutative covering projections. Suppose that it is following commutative diagram:

$$\begin{array}{ccc} \widetilde{A} & \xrightarrow{\widetilde{f}} & \widetilde{B} \\ \uparrow & & \uparrow \\ A & \xrightarrow{f} & B \end{array}$$

and homomorphism $\overline{f} : G(\widetilde{B}|B) \to G(\widetilde{A}|A)$ which satisfies following conditions

$$\widetilde{f}(g \cdot a) = \overline{f}(g) \cdot \widetilde{f}(a).$$

A homomorphism $\overline{f}$ is called *Homomorphism of fundamental groups with respect to* $A \to \widetilde{A}$, $B \to \widetilde{B}$ *coverings*.

Example 5.17. *Homomorphism of fundamental groups of noncommutative torus*. Let $A = A_\theta = \mathbb{C}[u, v]$, $B = A_{\theta mn} = \mathbb{C}[u', v']$ be C^* algebras and $f : A \to B$ *- homomorphism defined by following way:

$$u \mapsto u'^m,$$
$$v \mapsto v'^n.$$

Let $A \to \widetilde{A}$, $B \to \widetilde{B}$ coverings defined by following way:

$$\widetilde{A} = A_{\theta/m'n'} = \mathbb{C}[\widetilde{u}, \widetilde{v}],\ \widetilde{B} = A_{\theta/mm'nn'} = \mathbb{C}[\widetilde{u'}, \widetilde{v'}],$$

$$u \mapsto \widetilde{u}^{m'},\ v \mapsto \widetilde{v}^{n'},\ u' \mapsto \widetilde{u'}^{mm'},\ v' \mapsto \widetilde{v}^{nn'}.$$

It is clear that $G(\widetilde{A}, A) \approx \mathbb{Z}_{m'} \times \mathbb{Z}_{n'}$, $G(\widetilde{B}, B) \approx \mathbb{Z}_{mm'} \times \mathbb{Z}_{nn'}$. Homomorphism $\widetilde{f} : \widetilde{A} \to \widetilde{B}$ is defined as:

$$\widetilde{u} \mapsto \widetilde{u'}^{m'}, \widetilde{v} \mapsto \widetilde{v'}^{n'}.$$

These homomorphisms satisfy conditions of definition 5.16. Direct checking shows that Homomorphism of fundamental groups with respect to above coverings is natural surjective homomorphism:

$$\overline{f} : \mathbb{Z}_{mm'} \times \mathbb{Z}_{nn'} \to \mathbb{Z}_{m'} \times \mathbb{Z}_{n'}.$$

5.3 Homotopy groups

Let us construct noncommutative generalization of homotopy groups π_n $(n \geq 0))$. It is well known that If $X = \coprod_{\iota \in I} X_\iota$ is disjoint union of connected components then $\pi_0(X) \simeq \bigcup_{\iota \in I} \{X_\iota\}$ and $C(X) \simeq \oplus_\iota C(X_\iota)$. This fact provides evident generalization, i.e if $A = \oplus_{\iota \in I} A_\iota$ and A_ι is not a direct sum of algebras $\forall \iota \in I$ then

$$\pi_0(A) \simeq \bigcup_{\iota \in I} \{A_\iota\} = \bigcup \mathrm{Hom}^0_{\mathbb{C}}(\mathbb{C}, A) \tag{33}$$

Generalization of π_1 can is described by other chapters of this article. Generalization of π_n can be given as generalization of following formula

$$\pi_{n+m}(X, x_0) = \pi_n(\Omega^m(X, x_0)), \tag{34}$$

where $\Omega^m(X, x_0)$ is the m times iterated loop space (see 13.6). Definition 13.23 supplies noncommutative generalization of the loop space. So noncommutative homotopy groups can be given as

$$\pi_n(A) = \pi_0(\Omega^n(A)), \tag{35}$$

or

$$\pi_n(A) = \pi_1(\Omega^{n-1}(A)). \tag{36}$$

where $\Omega^n(A)$ is noncommutative generalization of n-times iterated loop space. Formulas (35),(36) mean set theory functions only. Noncommutative loop space is described in 13.2.3. There are two versions of the noncommutative loop space:

1. Pointed version $\Omega(A)$. This version is applicable for pointed C^* algebras only (See definition 13.22). Pointed version supplies a group structure on

$$\mathrm{Hom}(B, \Omega(A)).$$

2. Unpointed version $\Omega_0(A)$ This version is applicable for any (pro-) C^* - algebra. Unpointed version does not supply a group structure.

If $A = C(X)$ then

$$\pi_0(\Omega(A)) = \pi_1(X)$$

$$\pi_0(\Omega_0(A)) = \pi_1(X)/[\pi_1(X), \pi_1(X)]$$

So if $\pi_1(X)$ is an Abelian group, then

$$\pi_0(\Omega(A)) = \pi_0(\Omega_0(A)).$$

Although pointed version supplies a group structure there are a lot of interesting C^* - algebras which do not support the point algebra structure. For example, if A_θ is a noncommutative torus then a nontrivial homomorphism

$$A_\theta \to \mathbb{C}$$

does not exist, i.e. noncommutative torus is not a pointed algebra. However there are noncommutative torus A_θ is not a pointed algebra. According to (35), (36) we can define $pi_1(A_\theta)$ by following two ways

$$\pi_n(A_\theta) = \pi_0(\Omega_0^n(A_\theta)), \tag{37}$$

or

$$\pi_n(A_\theta) = \pi_1(\Omega_0^{n-1}(A_\theta)). \tag{38}$$

Formula (37) does not supply group structure on $\pi_n(A_\theta)$ but formula supplies(38 it.

6 Abelian covering projections

6.1 Abelian fundamental group

Calculation of Galois groups can be very difficult problem. Often difficult problem is replaced by simplified one. For example *class field theory* [43] is a powerful tool calculation of Abelian Galois groups of field extension.

Definition 6.1. A finite covering is called an *Abelian covering* if its Galois group is an Abelian group.

Let A be a C^* - algebra. Denote by $\mathrm{AbCov}(A)$ a category such that:

1. Objects of $\mathrm{AbCov}(A)$ are Abelian covering projections;

2. Morphisms of $\mathrm{AbCov}(A)$ are covering morphisms (See definitition 5.11).

Definition 6.2. Let A be C^* - algebra. If we replace category $\mathrm{Cov}(A)$ by category $\mathrm{AbCov}(A)$ we obtain the *universal Abelian covering space* instead the universal covering space (See definition . Also we obtain Abelian fundamental group $\pi_{ab}(A)$ instead fundamental group $\pi_1(A)$.

6.2 Natural constructions of cyclic covering projection

Let X be a locally compact Hasdorff space and f is a discontinuous function on X which can be regarded as a operator $f \in B(\mathcal{L}^2(X))$ which commutes with all elements of $C(X)$. Let $A = C(X)\{f\}$ is a C^* - commutative subalgebra of $B(\mathcal{L}^2(X)$ generated by $C(X)$ and f. We have a continuous map, $C(A) \to X$.

Definition 6.3. Let us represent a circle $S^1 = U(\mathbb{C})$ as $\{z \in \mathbb{C} \mid |z| = 1\}$. A *natural generator of* $C(S^1)$ is a unitary element $u \in U(C(S^1))$ such that u is represented by function $z \in C(S^1)$.

Definition 6.4. Let $U(\mathbb{C}) = \{z \in \mathbb{C} \mid |z| = 1\}$. A *$n$ -th root* is a Borel-measurable function $\phi \in \ B_\infty(U(C(S^1))$ such that

$$(\phi(z))^n = z \ (\forall z \in U(\mathbb{C})). \tag{39}$$

Let $u \in U(C(S^1))$ be a natural generator. Denote by $B(\phi)$ an operator $\phi(u)$.

Definition 6.5. We say that the unitary operator $u \in B(H)$ *has full spectrum* if $\mathrm{sp}(u) = \{z \in \mathbb{C} \mid |z| = 1\}$.

Definition 6.6. Let A be a C^* - algebra, $A \to B(H)$ faithful representation and $v \in U(B(H))$, $v^n \in A^+$, $v^i \notin A^+$ $(i < n)$. Let B be a minimal C^* - algebra which contains following operators:

1. $v^i a$, $(\forall i \in \mathbb{Z},\ \forall a \in A)$;
2. $a v^i$, $(\forall i \in \mathbb{Z},\ \forall a \in A)$;

Denote by $A\{v\} = B$. C^* - algebra $A \to A\{v\}$ is said to be a *n - th root multiplier extension*.

Example 6.7. Let A be a C^* - algebra, $u \in U(A^+)$ has full spectrum and $\nexists v \in U(A^+)$, $v^n = u$. Let $\rho : A \to B(H)$ is faitful representation, ϕ is a n - th root and $v = \phi(\rho(u))$. Then $v^n = \rho(u)$. We have a injective homomorphism $A \to A\{v\}$. Let $\alpha \in \mathrm{Aut}(A\{v\})$ be an automorphism defined by following way:

$$v \mapsto e^{\frac{2\pi i}{n}} v. \tag{40}$$

It is clear that $\alpha^n = \mathrm{Id}_{A\{v\}}$. This automorphism is said to be a *natural n - cyclic automorphism*. It defines action of $\mathbb{Z}_n$ on $A\{v\}$ and $(A\{v\})^{\mathbb{Z}_n} = A$. This action is said to be *the natural cyclic action*.

Lemma 6.8. *Let A be a C^* - algebra, $u \in U(A^+)$, has full spectrum, ϕ, ψ are n -th roots, then it is isomorphism $A\{\phi(u)\} \otimes \mathcal{K} \approx A\{\psi(u)\} \otimes \mathcal{K}$*

Proof. It is clear that $A\{\phi(u)\} \otimes \mathcal{K} \approx (A \otimes \mathcal{K})\{\phi(u) \otimes 1\}$. We have following isomorphism:

$$(A \otimes \mathcal{K})\{\phi(u) \otimes 1\} \approx (A \otimes \mathcal{K})\{\psi(u) \otimes 1\},$$

$$\phi(u) \otimes 1 \leftrightarrow \psi(u) \otimes B(\phi\psi^{-1}).$$

Definition of $B(\phi\psi^{-1})$ is contained in 6.4 □

Lemma 6.9. *Let $f : A \to A\{v\}$ be a n -th root multiplier extension. Then f is a $\mathbb{Z}_n$ - Galois extension, and $A\{v\}$ is a free finitely generated left and right A - module.*

Proof. It is clear that $u = v^n \in A$ is unitary element. Since u and v are unitary elements there are following isomorphisms:

$$C(u) \approx C(S^1);\ C(v) \approx C(S^1).$$

Map $C(u) \to C(v)$ corresponds to n listed covering projection $p : S^1 \to S^1$ of the circle and $\mathbb{Z}_n$ is its group of covering transformations. There are functions $a_1, ..., a_n, b_1, ..., b_n$ such that:

$$\sum_{i=1}^{n} a_i b_i = 1_{S^1};$$

and if $g \in \mathbb{Z}_n$ is a not trivial element of covering transformation group then

$$\sum_{i=1}^{n} a_i g b_i = 0.$$

Functions $a_1, ..., a_n, b_1, ..., b_n$ can be regarded as elements of $A[v]$. So according to 4.4 $A \to A[v]$ is a Hopf - Galois extension. Also $A[v]$ is a free finitely generated A - module, generated by v, ... v^{n-1}. A generator of $\mathbb{Z}_n$ acts on $A[v]$ by following way:

$$v \mapsto e^{\frac{2\pi i}{n}} v.$$

It is clear that:

$$A\{v\} \approx A \oplus vA \oplus ... \oplus v^{n-1}A,$$

$$A\{v\} \approx A \oplus Av \oplus ... \oplus Av^{n-1}.$$

□

Example 6.10. Let A be a C^* - algebra, $x \in K^1(A)$ a generator of infinite order, $u \in U((A \otimes \mathcal{K})^+)$ is representative of x, i.e. $x = [u]$. If $\phi \in B_\infty(U(\mathbb{C}))$ is an n - th root, then $\phi(u) \notin M(A \otimes \mathcal{K})$. So $A \otimes \mathcal{K} \to (A \otimes \mathcal{K})\{\phi(u)\}$ is a $\mathbb{Z}_n$ - Galois extension.

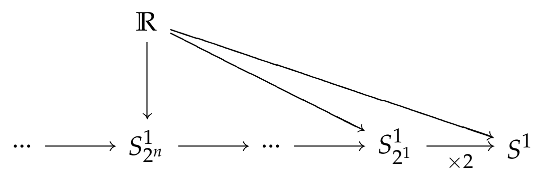
Following proposition is used in below constructions.

Proposition 6.11. *[28] If A is a unital local C^*, then $U_1(A)_0$ is generated algebraically by $\{e^{ix} : x = x^* \in A\}$. If B is also unital local C^* - algebra and $\phi : A \to B$ is surjective and unital, then $\phi(U_1(A)_0) = U_1(B)_0$.*

7 Noncommutative generalization of $\mathbb{R} \to S^1$ covering

7.1 $\mathbb{R} \to S^1$ covering

$\mathbb{R} \to S^1$ *covering*. Let $p : \mathbb{R} \to S^1$ well known infinitely listed covering [34]. No function $f \in C_0(\mathbb{R})$, $f \neq 0$ can be obtained from function $g \in C_0(S^1)$. However it is another alternative construction which is modification universal object. Let us consider category of all coverings of circle.



We assume that all coverings of above coverings are two listed coverings. Symbol $S^1_{2^n} \approx S^1$ means that that covering degree of initial circle S^1 is equal to 2^n. It is natural bijection between $C(S^1)$ and $\{f :\in C[-2^n\pi, 2^n\pi], f(-2^n\pi) = f(2^n\pi)]\}$. Any $f \in C_0(\mathbb{R})$ can be regarded as limit of functions supported on intervals $[-2^n\pi, 2^n\pi]$, $n \to \infty$. More precisely for $f \in C_0(\mathbb{R})$ it is following sequence $f_{2^n} \in C_0(\mathbb{R})$:

$$f_{2^n}(x) = \begin{cases} f(x) - f(2^n\pi) + (x - 2^n\pi)\frac{f(x+2^n\pi)-f(x-2^n\pi)}{2^{n+1}\pi}; \; x \in [-2^n\pi, 2^n\pi] \\ 0; \; x \notin [-2^n\pi, 2^n\pi] \end{cases} \tag{41}$$

It is evident that sequence f_{2^n} is norm convergent to f. Now let us note that

$$L^2(\mathbb{R}) = \bigoplus_{k\in\mathbb{Z}} L^2([2\pi k, 2\pi(k+1)]), \tag{42}$$

where $\oplus$ means Hilbert direct sum. Otherwise f_{2^n} can be naturally identified with element $\overline{f}_{2^n} \in S^1_{2^n}$ It is clear that there are following natural isomorphisms of Hilbert spaces

$$L^2([2\pi k, 2\pi(k+1)]) \approx L^2([2\pi n, 2\pi(n+1)]) \; \forall k, n \in \mathbb{Z}.$$

Let us also define action $\overline{f}_{2^n} \in S^1_{2^n}$ on $L^2(\mathbb{R})$ as action of f_{2^n}. It is clear that $\overline{f}_{2^n}$ trivially acts on $L^2([-2\pi k, 2\pi(k+1)])$ $(n < k \bigvee n > k+1)$.

7.2 Generalization of $\mathbb{R} \to S^1$ covering

Algebra $C(S^1)$ satisfies conditions of definition 6.9. The $\mathbb{R} \to S^1$ covering can be generalized on any C^* algebra which satisfies conditions of definition 6.9. Let A and $u \in U(A)$ be such algebra and its unitary element which satisfy to conditions of definition 6.9. Algebra A faithfully acts on Hilbert space H. Let us consider following sequence of multiplier extensions (See 6.6) :

$$A \to A\{v_2\} \to A\{v_4\} \to ... \to A\{v_{2^n}\} \to ... \tag{43}$$

where $v_{2^n}^{2^n} = u$.
Above diagram is noncommtative analogue of diagram considered at 7.1. Indeed $C(S^1) \approx C(u)$ and $C(S^1_{2^n}) \approx C(v_{2n})$. Let us define action of $C_0(\mathbb{R})$ on Hilbert sum $\widetilde{H} = \bigoplus_{n\in\mathbb{Z}} H_n$. First of all note that $C(v_{2^k})$ acts on $\bigoplus_{-2^k \leq n < 2^k - 1} H_n$. Suppose that $C(v_{2^k})$ acts trivially on H_n if $n < -2^k \bigvee n \geq 2^k$. This action can be continued whole sum $\widetilde{H}$. Let $f \in C(\mathbb{R})$ any function and sequence f_n is defined by equation 41. On can define functions $\overline{f}_{2^n} \in C(S^1_{2^n}) = C(v_{2^n})$. So l f_{2^n} $\forall n \in \mathbb{N}$ defines bounded operator $B(f_{2^n}) \in B(\widetilde{H})$. Sequence $B(f_{2^n})$ is norm convergent. By $B(f)$ denote its limit. Denote $B(a) \in B(\widetilde{H})$ $a \in A$ bounded operator which acts on every component of Hilbert sum as well as a acts on H.

Definition 7.1. Let $B \in B(\widetilde{H})$ be norm completion of algebra generated by elements $B(a)B(f)$ and $B(f)B(a)$, where $a \in A$ and $f \in C(\mathbb{R})$. This algebra is called *Noncommutative generalization of* $\mathbb{R} \to S^1$ *covering*.

Remark 7.2. This construction can be generalized. Suppose that there are two elements $u_1, u_2 \in A$ which satisfy conditions 6.9. We can define two actions of $C(\mathbb{R})$ on $\widetilde{H}$. Let us distinguish these actions for clarity. Action of $C(\mathbb{R})_1$, $C(\mathbb{R})_2$ is constructed by usage of elements u_1 and u_2 respectively. Norm completion of algebra generated by $B(f_1)B(f_2)B(a)$,$B(f_2)B(f_1)B(a)$, $B(f_1)B(a)B(f_2)$, $B(f_2)B(a)B(f_1)$, $B(a)B(f_1)B(f_2)$, $B(a)B(f_2)B(f_1)$, where $a \in A$, $f_1 \in C(\mathbb{R})_1$ $f_2 \in C(\mathbb{R})_2$, can be regarded as generalization of covering of torus be plane. Similarly covering of n torus by R^n can be generalized.

Example 7.3. *Infinite covering of noncommutative torus*. Algebra A_θ of noncommittal torus has two unitary elements u, v which satisfy conditions of definition 6.9. So infinite generalization of covering by plane can be constructed. Since v satisfies condition 6.9 one can construct such sequence $v_2, v_2^*, ..., v_{2^n}, v_{2^n}^*$ that $v_{2^n}^{2n} = v$, $v_{2^n}^{*2n} = v^*$; $\forall n \in \mathbb{N}$. Elements of this sequence satisfy following conditions:

$$uv_{2^n} = e^{2\pi i(\theta+k)/2^n} v_{2^n} u,$$

where $k \in \mathbb{Z}$ is arbitrary integer number. Here we set $k = 0$. In this case $uv_{2^n} = e^{2\pi i\theta/2^n} v_{2^n} u$. Sequence v_{2^n} induces sequence $B(f_{2^n}) \in B(\widetilde{H})$ for all $f \in C_0(\mathbb{R})$. $B(f) \in B(\widetilde{H})$ is norm limit of $B(f_{2^n})$ (See 7.2). Operators $B(u)$ and $B(f_{2^n})$ satisfy following condition.

$$B(u)B(f_{2^n}) = e^{2\pi i\theta/2^n} B(f_{2^n})B(u).$$

Since $B(f)$ is norm limit of $B(f_{2^n})$ we have.

$$B(u)B(f) = B(f)B(u).$$

From previous equation it follows that algebra generated by elements $B(u)$ and $B(f)$ $\forall f \in C_0(\mathbb{R})$ is commutative. So its norm completion is also commutative. One can check that this algebra is isomorphic to $C_0(S^1 \times \mathbb{R})$. Generalization of infinite covering of $C_0(S^1 \times \mathbb{R})$ is $C_0(\mathbb{R}^2)$. This generalization coincides with commutative covering. So infinite covering of noncommutative torus is commutative plane.

8 Generalization of Hurewicz homomorphism

Notion of Hurewicz homomorphism was initially appeared in algebraic topology and then generalized in several directions. This chapter is devoted to generalization related to theory of C^* - algebras. First of all let us remind some notions of algebraic topology [34]. Let X be topological space, $x_0 \in X$ is base point, $\pi_n(X, x_0)$, $H_n(X)$ are n-th ($n \in \mathbb{N}$) are n-th homotopy group and singular homology group respectively. Then $\forall n \in \mathbb{N}$ there is natural homomorphism $\phi_n : \pi_n(X, x_0) \to H_n(X)$. This homomorphism is named *Hurewicz homomorphism*. This notion can be naturally extended for any generalized homology theory $h_*(-)$ (See [2]) such that $h^*(S^n) \approx \mathbb{Z}$. If K^1 is K - homology theory of C^* algebras then $K_{n \bmod 2}(C(S^n)) \approx \mathbb{Z}$. So we have natural transformation $\pi_*(-) \to K^*(C(-))$ which can be regarded as generalization of Hurewicz homomorphism. In case of fundamental group we have a natural homomorphism

$$\pi_1(X) \to K^1(C(X)),\ [f] \mapsto K^1(f)(u) \tag{44}$$

where $u \in K^1(C(S^1)) \approx \mathbb{Z}$ is a generator $f : S^1 \to X$ is a representative of $[f] \in \pi_1(X)$. We would like supply a noncommutative analogue of this homomorphism.

8.1 Universal coefficient theorem

For any C^* - algebra A there is a natural homomorphism

$$\gamma : KK_1(A, \mathbb{C}) \to \mathrm{Hom}(K_1(A), K_0(\mathbb{C})) \approx \mathrm{Hom}(K_1(A), \mathbb{Z})$$

which is the adjoint of following pairing

$$KK(\mathbb{C}, A) \otimes KK(A, \mathbb{C}) \to KK(\mathbb{C}, \mathbb{C}). \tag{45}$$

If $\tau \in KK^1(A, \mathbb{C})$ is represented by extension

$$0 \to A \to D \to \mathbb{C} \to 0$$

then γ is given connecting maps ∂ in the associated six-term exact sequence of K theory

$$\begin{array}{ccccc}
K_0(A)) & \longrightarrow & K_0(D) & \longrightarrow & K_0(\mathbb{C}) \\
\partial \uparrow & & & & \downarrow \partial \\
K_1(A) & \longleftarrow & K_1(D) & \longleftarrow & K_1(\mathbb{C})
\end{array}$$

If $\gamma(\tau) = 0$ for an extension τ then the six-term K-theory exact sequence degenerates into two short exact sequences

$$0 \to K_i(A) \to \mathcal{K}_i(D) \to K_i(\mathbb{C}) \to 0 \ (i = 0, 1)$$

and thus determines an element $\kappa(\tau) \in \mathrm{Ext}^1(K_*(A), K_*(\mathbb{C})$. In result we have a sequence of abelian group homomorphisms

$$\mathrm{Ext}^1(K_0(A), K_0(\mathbb{C})) \to KK^1(A, \mathbb{C}) \to \mathrm{Hom}(K_1(A), K_0(\mathbb{C}))$$

such that composition of the homomorphisms is trivial. Above sequence can be rewritten by following way

$$\mathrm{Ext}^1(K_0(A), \mathbb{Z}) \to K^1(A) \to \mathrm{Hom}(K_1(A), \mathbb{Z})). \tag{46}$$

If G is an abelian group that

$$\mathrm{Ext}^1(G, \mathbb{Z}) = \mathrm{Ext}^1(G_{tors}, \mathbb{Z})$$

$$\mathrm{Hom}(G, \mathbb{Z}) = \mathrm{Hom}(G/G_{tors}, \mathbb{Z})).$$

From (46) it follows that $K^1(A)$ depends on $K_0(A)_{tors}$ and $K_1(A)/K_1(A)_{tors}$. We say that dependence on $K_0(A)_{tors}$ is a *torsion special case* and dependence on $K_1(A)/K_1(A)_{tors}$ is a *free special case*.

8.2 Free special case

Example 8.1. It is well known that $\pi_1(S^1) \approx \mathbb{Z}$ and $K^1(C(S^1)) \approx \mathbb{Z}$. Natural homomorphism $\pi_1(S^1) \to K^1(C(S^1))$ given by (44) is an isomorphism. This isomorphism can be regarded by algebraic way. From 46 it follows that $K_1(C(S^1)) \approx \mathbb{Z}$. Let $u \in U(C(S^1))$ is such that $[u] \in K_1(S^1)$ is a generator of $K_1(S^1)$. Let $C(S^1) \to B(H)$ be a full represetation and ϕ is an n - th root (See 6.4). If $v = \phi(u) \in B(H)$ then $v^n = u$ and $v \notin C(S^1)$. Element u has full spectrum, so we have n - th root multiplier extension $C(S^1) \to C(S^1)\{v\}$ (See 6.6). According to 7 we can construct generalisation of infinite covering projection.

8.2. Above construction can be generalized. Let A be a C^* - algebra such that $K^1(A) \approx G \oplus \mathbb{Z}$. From (46) it follows that

$$K_1(A) = G' \oplus \mathbb{Z}. \tag{47}$$

Let $u \in U((A \otimes \mathcal{K}))$ is such that $[u] \in K_1(A)$ is a generator of second summand of (47). From (46) it follows that there is $u \in U(A)$ such that $[u] \in K_1(A)$ is element of infinite order and $[u]$ is not divisible. Suppose that u has full spectrum (See 6.5). Let $A \otimes K \to B(H)$ and $v \in B(H)$ is such that $v^n = u$ ($n > 1$) then we can construct noncommutative finite covering projections $A \to A\{v\}$ and generalization of infinite covering projection.

Example 8.3. It is known that S^3 is homeomorphic to $SU(2)$, $K_1(C(SU(2))) \approx \mathbb{Z}$ and $K_1(C(SU(2)))$ is generated by unitary $u \in U(C(SU(2) \otimes \mathbb{M}_2(\mathbb{C}))$. Element u can be regarded as the natural map $SU(2) \to \mathbb{M}_2(\mathbb{C})$ and u has full spectrum. Let ϕ be a n - th root, and $v = \phi(u)$. Denote by A a C^* - algebra $C(SU(2)) \otimes \mathbb{M}_2(\mathbb{C})$. Both A and $A\{v\}$ are continuous trace algebras. Let $\rho \in \mathrm{Irr}(A\{v\})$ be a irreducible representation of B. Then $V = \rho(v)$ is a 2×2 unitary matrix. The unordered pair of eigenvalues λ_1, λ_2 of

V is an invariant of ρ, i.e. unitary equivalent representations have equal values of these invariant. Group Z_n changes this invariant by following way:

$$\lambda_1 \mapsto c\lambda_1,\ \lambda_2 \mapsto c\lambda_2,\ c = e^{\frac{2\pi k}{n}},\ k \in \mathbb{N}.$$

Group Z_n acts freely on this invariant and therefore Z_n acts freely on $\widehat{A\{v\}}$. So $\widehat{A\{v\}} \to \hat{A}$ is a n - listed covering projection. Since $\pi_1(SU(2)) = \pi_1(S^3)$ is trivial $\hat{B} = \amalg_{i=1,\dots,n} S^3$ is a disjoint union of n copies of S^3.

Remark 8.4. Addition of discontinuous function can transform connected space to disconnected one. Topological fundamental group concerns with connected covering projections. In example 8.1 we have connected covering space, in example 8.3 we have disconnected covering space.

Example 8.5. Let A_θ be a noncommutative torus, $K^1(A_\theta \approx \mathbb{Z}^2)$ Let $u, v \in U(A)$ be reprezentatives of generators of $K^1(A_\theta)$ Both u and v has full spectrum. We can construct generalization of universal covering projection.

Example 8.6. *Coverings of noncommutative 3D Sphere* Algebra of complex functions of 3D sphere could be generated by four real valued functions x_1,...,x_4 those satisfy to following equations:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1. \tag{48}$$

If we introduce complex valued functions $\alpha = x_1 + ix_2$, $\beta = x_3 + ix_4$ then when we can replace (48) by the following equation:

$$\alpha\alpha^* + \beta\beta^* = 1. \tag{49}$$

Very interesting involutive noncommutative algebra is considered in [44]. It is generated by two elements α, β and satisfies to following relations.

$$\alpha^*\alpha + \beta^*\beta = 1;\ \alpha\alpha^* + q^2\beta\beta^* = 1;\ \alpha\beta - q\beta\alpha = 0;\ \alpha\beta^* - q\beta^*\alpha = 0;\ \beta^*\beta = \beta\beta^*, \tag{50}$$

where q is a real number and $0 < q \leq 1$.
By $C(SU_q(2))$ denote C^* which satisfy above equation. It is clear that if we suppose that $q = 1$ then this algebra is commutative and it satisfies to relations (48). If $q \approx 1$ then algebra $C(SU_q(2))$ could be considered as noncommutative approximation of algebra $C(S^3)$ of continuous complex valued functions on 3D sphere. $C(SU_q(2))$ admits the structure of spectral triple[42]. It is well known that 3D sphere is simply connected. So if $q = 1$ then $C(SU_q(2))$ no nontrivial finite coverings. However if $q = 1$ then it is such unitary element $u \in U(C(SU_q(2))$ than $[u] \in K^1(C(SU_q(2))$ is not trivial and has infinite period. Element u comply conditions of definition 6.6. So one can construct construct cyclic covering $C(SU_q(2)) \to B$ where B is generated over a by such element v that $v^n = u$ and we have noncommutative covering projections $C(SU_q(2)) \to C(SU_q(2))\{v\}$. It is not known is this projection connected.

Remark 8.7. If $q \approx 1$ then algebraic properties of $C(SU_q(2))$ are very close to algebraic properties of commutative algebra $C(S^3)$. However these algebras are principally different. First one does not have nontrivial coverings but second one has them. Perhaps this fact is relevant to structure of the Universe. In some models space of the Universe is $C(S^3)$. Since $C(SU_q(2))$ is close $C(S^3)$ it is reasonably suppose that Universe space correspond to algebra $C(SU_q(2))$. Since former algebra has nontrivial coverings this fact can occur new cosmological properties.

8.3 Torsion special case

Example 8.8. Let $f : S^1 \to S^1$ be a n listed covering projection of the circle, C_f is the (topological) mapping cone of f. $C(f) : C(S^1) \to C(S^1)$ is a corresponding *- homomorphism of C^* algebras ($u \mapsto u^n$), where $u \in U(C(S^1))$ is such that $[u] \in K_1(C(S^1))$ is a generator. Algebraic mapping cone [28] $C_{C(f)}$ of $C(f)$ corresponds to the topological space C_f. $C_{C(f)}$ is an algebra of continuous maps $f[0,1) \to U(\mathbb{C})$ such that:

$$f(0) = \sum_{k \in \mathbb{Z}} a_k u^{kn},\ a_k \in \mathbb{C}.$$

It is well known that $\pi_1(C_f) \approx \mathbb{Z}_n$ and $K^0(C_f) = K_0(C(C_f)) \approx \mathbb{Z}_n$. From (46) it follows that $K^1(C(C_f)) \approx \mathbb{Z}_n$ A map $v = (x \mapsto u)$ ($\forall x \in [0,1]$) has full spectrum and v is not a multiplier of $C(C_f)$. However $v^n \in M(C_{C(f)})$. Let $M(C_{C(f)}) \to B(H)$ be a faithful repersentation. Homomorphism $C_{C(f)} \to C_{C(f)}\{v\}$ is a $\mathbb{Z}_n$ - Galois extension. It a is indeed a noncommutative covering projection which corresponds to (connected) n - listed universal topological covering projection of C_f. So $\pi_1(C_f, x_0) = \mathbb{Z}_n$. Otherwise v can be regarded as representative of $K_1(Q^s(C_{C(f)}))$. However $K^1(Q^s(C_{C(f)})) \approx K^0(C_{C(f)}) \approx Z_n$.

8.9. Above construction can be generalized. Let A be a C^* - algebra such that $K_0(A) = G \oplus \mathbb{Z}_n$, where G is an abelian group. From (46) it follows that $K^1(A) \approx G' \oplus \mathbb{Z}_n$ Let $Q^s(A) = (M(A \otimes \mathcal{K})/(A \otimes \mathcal{K})$ be the stable multiplier algebra of C^* - algebra A. Then from [28] it follows that $K_1(Q^s(A)) = K_0(A)$ Let $x \in K^1(Q^s(A))$ be a generator of direct summand $\mathbb{Z}_n \subset K^1(Q^s(A))$ and $u \in U(Q^s(A))$ is representative of x, i.e $x = [u]$. Suppose that u has full spectrum (See 6.5). Let ϕ be a n - th root such that $\phi(u^n) = u$. Let $p : (M(A \otimes \mathcal{K})) \to U(M(A \otimes \mathcal{K}))/(A \otimes \mathcal{K})$ be a natural surjective *- homomorphism. Since $n[u] = [u^n] = 0$ then from [28] it follows that there is an unitary $v \in U(M(A \otimes \mathcal{K}))$ such that $p(v) = u^n$. Suppose that v has full spectrum. Now we can construct multiplier extension $A \otimes \mathcal{K} \to (A \otimes \mathcal{K})\{\phi(v)\}$.

8.10. If $0 \to J \to A \to A/J \to 0$ is an exact sequence of C^* algebras then there is following sequence (see [28]):

$$K_1(J) \to K_1(A) \to K_1(A/J) \xrightarrow{\partial} K_0(J) \to K_0(A) \to K_0(A/J).$$

Where ∂ is defined by following way. Let $u \in \mathrm{GL}_n(A/J)$ and let $w \in \mathrm{GL}_{2n}(A)$ be a lift of $\mathrm{diag}(u, u^{-1})$. Define $\partial([u]) = [wp_nw^{-1}] - [p_n]$. The map ∂ is called the *index map*. The reason is the following. Suppose A is a unital C^* - algebra and u is a unitary in $\mathbb{M}_n(A/J)$. If u lifts to a partial isometry $v \in \mathbb{M}_n(A)$, then $\text{diag}(u, u^{-1})$ lifts to the unitary

$$w = \begin{pmatrix} v & 1 - vv^* \\ 1 - v^*v & v^* \end{pmatrix}.$$

so $\partial([u]) = [wp_nw^{-1}] - [p_n] = \text{diag}(vv^*, 1 - v^*v] = [1 - v^*v] - [1 - vv^*]$. In the special case we have following exact sequence

$$0 \to A \otimes \mathcal{K} \to M(A \otimes \mathcal{K}) \to M(A \otimes \mathcal{K})/(A \otimes \mathcal{K}) \to 0.$$

It is shown in [28] that $K_0(M(A \otimes \mathcal{K})) = K_1(M(A \otimes \mathcal{K})) = 0$. So $\partial : K_1(M(A \otimes \mathcal{K})/(A \otimes \mathcal{K}) \to K_0(A \otimes \mathcal{K} = Q^s(A))$ is a isomorphism.

Example 8.11. Let O_n be the C^* - algebra generated by n isometries $s_1, ..., s_n$ with $s_i^* s_i = 1$, $s_i s_i^* = p_i$, and $p_1 + ... + p_n = 1$. The O_n was sudied by Cuntz [8]. In $K_0(O_n)$ we have $[p_1] = ...[p_n] = [1]$, so $n[1] = [p_1] + ... + [p_n] = [1]$, i.e. $n[1] = [1]$, i.e. $(n-1)[1] = 0$. It is shown in [9] that $K_0(O_n) = \mathbb{Z}_{n-1}$ with [1] as generator. Let

$$v = \begin{pmatrix} s_1 & 0 & 0... \\ 0 & 1 & 0... \\ .... & & \end{pmatrix} \in M(O_n \otimes \mathcal{K}) = M^s(O_n).$$

Element v_1 is a isometry and

$$v_1^* v - 1_1 = \begin{pmatrix} p_1 - 1_{O_n} & 0 & 0... \\ 0 & 0 & 0... \\ .... & & \end{pmatrix} \in O_n \otimes \mathcal{K}.$$

it is representative of unitary element $\overline{v} \in Q^s(O_n)$ since

$$v_1^* v_1 = 1; \; v_1^* v_1 - 1 \in O_n \otimes \mathcal{K}.$$

From index map it follows that $[\overline{v}] \in K_1(Q^s(O_n))$ is a generator. From $K_1(Q^s(O_n)) = \mathbb{Z}_n$ it follows that $(n-1)[\overline{v}] = [\overline{v}^{n-1}] = 0$. So $\overline{v}^{n-1}$ can be lifted to unitary element $w \in M^s(O_n)$. Let $\phi \in B_\infty(U(\mathbb{C}))$ be $n-1$ - th root. Then $O_n \otimes \mathcal{K} \to (O_n \otimes \mathcal{K})\{\phi(w)\}$ is a $\mathbb{Z}_{n-1}$ - Galois extension.
Let us provide explicit expression for $M^s(O_n)$ representative of $[\overline{v}_1^n]$

Denote by $v_2, v_3, ...v_n \in M^s(O_n)$ following elements

$$v_2 = \begin{pmatrix} 1 & 0 & 0 & ... \\ 0 & s_2 & 0 & ... \\ 0 & 0 & 1 & ... \\ .... & & & \end{pmatrix}, \; v_3 = \begin{pmatrix} 1 & 0 & 0 & ... \\ 0 & 1 & 0 & ... \\ 0 & 0 & s_3 & ... \\ .... & & & \end{pmatrix}, \; ... \; v_n = ... \begin{pmatrix} 1 & 0 & 0 & ... & 0 & ... \\ 0 & 1 & 0 & ... & 0 & ... \\ 0 & 0 & 1 & ... & 0 & ... \\ ... & & & & & \\ 0 & 0 & 0 & ... & s_n & ... \end{pmatrix}.$$

It is clear that $[\overline{v}_1] = [\overline{v}_2] = ... = [\overline{v}_n] \in Q^s(O_n)$. So following partial isometry

$$u = \begin{pmatrix} s_1 & 0 & 0 & ... & 0 & 0 & ... \\ 0 & s_2 & 0 & ... & 0 & 0 & ... \\ 0 & 0 & s_3 & ... & 0 & 0 & ... \\ ... \\ 0 & 0 & 0 & ... & s_{n-1} & 0 & ... \\ 0 & 0 & 0 & ... & 0 & 1 & ... \end{pmatrix}.$$

is such that $[\overline{u}] = (n-1)[v_1] = 0$. So $\overline{u}$ should be lifted to $M^s(O_n)$. Following unitary element w is the lift of u

$$w = \begin{pmatrix} s_1 & s_2 & s_3 & ... & s_{n-1} & s_n & 0 & ... \\ 0 & s_2 & s_3 & ... & s_{n-1} & s_n + s_1 & 0 & ... \\ 0 & 0 & s_3 & ... & s_{n-1} & s_n + s_1 + s_2 & 0 & ... \\ ... \\ 0 & 0 & 0 & ... & s_{n-1} & s_n + s_1 + ... + s_{n-2} & 0 & ... \\ 0 & 0 & 0 & ... & 0 & 1 & 0 & ... \\ ... \end{pmatrix}.$$

9 Universal covering space and fundamental group

9.1. The algebraic topology contains following well known theorems:

Theorem 9.2. *A connected locally path-connected space X has a simply connected covering space if and only if X is semilocally 1-connected.*

Theorem 9.3. *Any universal covering space of a connected locally path-connected is semilocally 1-connected space is simply connected.*

So if a topological space X satisfies the conditions of theorems 9.2, 9.3 then $\pi_1(X) = G(\widetilde{X}|X)$. Noncommutative generalizations of these conditions do not exist. However some C^* - algebras have analogue of an universal covering space. Following text contains definition of the universal covering space.

9.4. Let A be a C^* - algebra, $A \to A_i$ is a set of all finite covering projections. Let us introduce a category $\mathcal{C}_A$. Objects of $\mathcal{C}$ are pairs (B, G) where B is a C^* - algebra, G is a group which acts on B and following conditions are satisfied:

1. If $A \to A_i$ is a finite covering then there are both an injective *-homomorphism $f_i : A_i \to M(B)$ and surjective group homomorphism $h_i : G \to G(A_i|A)$ such that $g(f_i(a)b) = (f_i(h_i(g)))(gb)$.
2. Let $f_{12} : A_1 \to A_2$ be an A - covering morphism and $h_{12} : G(A_2|A) \to G(A_2|A_1)$ is a corresponding group homomorphism. Then we have:

$$f_1 = f_2 \circ f_{12};\ h_1 = h_{12} \circ h_2.$$

3. $\bigcap \ker(G \to G(A_i|A))$ is a trivial group.

A $\mathcal{C}_A$ - morphism from (B_1, G_1) to (B_2, G_2) contains an injective *- homomorphism $\phi : B_2 \to B_1$ and a surjective group homomorphism $\psi : G_1 \to G_2$ such that following conditions are satisfied:

$$\phi(ga) = \psi(g)\phi(a),\ \forall a \in B_2,\ \forall g \in G_2;$$

Definition 9.5. Let A be a C^* - algebra. Suppose that a category $\mathcal{C}_A$ is not empty and $(\widetilde{A}, G)$ is the final object of $\mathcal{C}_A$. Then $\widetilde{A}$ is said to be *the universal covering space of A*, G is said to be *the fundamental group of A*, $A \to M(\widetilde{A})$ is said to be *the universal covering projection of A*. Denote by $\pi_1(A) = G$ the fundamental group of A.

10 Generalization of arbitrary infinite covering

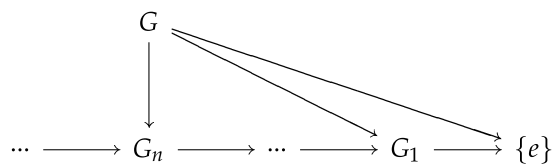
Here we would like construct generalization of arbitrary infinite covering. This construction is analogical to commutative infinite covering. So first of all algebraic construction of commutative infinite covering will be constructed.

10.1 Commutative infinite covering from algebraic viewpoint

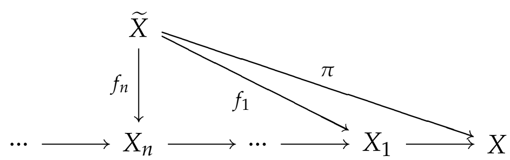
Let (X, x_0) be pointed topological space space, $\pi(\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is infinitely listed covering, $G = G(\widetilde{X}, X)$ is covering group. According to GNS Construction [29] C^* - algebra $C(X)$ has a faithful representation, i.e. $C(X)$ is isometrically isomorphic to C^*-algebra of operators on a Hilbert space H. Here full representation of $C(\widetilde{X})$ on Hilbert sum

$$\widetilde{H} = \oplus_{g \in G(\widetilde{X}, X)} H_g;\ H_g \approx H\ (\forall g \in G(\widetilde{X}, X)) \tag{51}$$

will be constructed. Let $U \in \widetilde{X}$ be connected fundamental domain i.e. U is open, limitation $\pi|_U$ is injective map, and $\pi(U) \in X$ is dense subset. Suppose that $\widetilde{x}_0 \in U$. Group $G(\widetilde{X}, X)$ acts on $\widetilde{X}$. Group G acts on $\widetilde{X}$ and gU is fundamental domain $\forall g \in G$. Denote by A'' bicommutant of C^* - algebra A [29]. Any faithful action of A on Hilbert space induces faithful action of A'' on same Hilbert space. Since $\pi(U)$ is dense in X we have $C(\pi(U))'' = C(X)''$. Set $\widetilde{U} = \pi^{-1}(\pi(U)) = \bigcup_{g \in G} gU$ is dense open subset of $\widetilde{X}$, $C(\widetilde{X})'' \approx C\widetilde{U}'' \approx \bigoplus_{g \in G} C(gU)''$. Otherwise $\bigoplus_{g \in G} C(gU)''$ acts on Hilbertian sum $\widetilde{H} = \bigoplus_{g \in G} H_g$, where $H_g \approx H\ \forall g \in G$. So $C(\widetilde{X})$ have faithful representation on $\widetilde{H} = \bigoplus_{g \in G} H_g$. Action of $\widetilde{a} \in C(\widetilde{X}$ is defined by following way. Element $\widetilde{a}$ is continuous function on $\widetilde{X}$. Its limitation $\widetilde{a}|_{gU}, (g \in G)$ is element of $C(gU)$, $\widetilde{a} \in C(gU)''$. So $\widetilde{a}|_{gU}$ acts on H_g. Action of $\widetilde{a}$ on $\widetilde{H} = \bigoplus_{g \in G} H_g$ is componentwise action of $\widetilde{a}|_{gU}$ on $H_g\ \forall g \in G$. Let us consider approximation of this action by actions obtained by finite coverings. Suppose that G can be included into following diagram of surjective group homomorphism:



where G is finite $\forall n \in \mathbb{N}$. Suppose that $\bigcap_{n \in \mathbb{N}} \ker(G \to G_n) = \{e\}$. This diagram induces following of coverings.



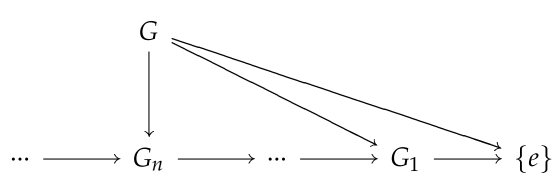
where $G(X_n|X) \approx G_n$, maps $p_n : X_n \to X \; \forall n \in \mathbb{N}$ are finite coverings.
Let $g_1, ..., g_k$ be all elements of G. G_n is quotient group of G. Let us select for all $g_i \in G_n$ such representative $\widetilde{g}_i$ that set $\overline{\bigcup_{i=1,...,m} \widetilde{g}_i U}$ is connected. Set $p(U_n) = \bigcup_{i=1,...,m} \widetilde{g}_i U$ is dense open subset of X_n. Slight modification of previous speculations shows that $C(X_n)$ have full representation on direct sum $\bigoplus H_{\widetilde{g}_i}$ $(i = 1, ..., m)$. Since $\bigoplus H_{\widetilde{g}_i}$ $(i = 1, ..., m) \subset \widetilde{H}$ $C(X_n)$ acts on $\widetilde{H}$. Let us select such fundamental domains $U_i \in X$, $i \in \mathbb{N}$ which correspond to spaces X_i and $U_i \subset U_j$, $(i < j)$. This selection define actions of $C(X_i)$ $i \in \mathbb{N}$ and all these actions are compatible with homomorphisms $C(X_i) \to C(X_j)$. Let $A \in B(\widetilde{H})$ be norm completion of algebra generated by $C(X_i) \in B(\widetilde{H})$. $C(X) \in A$ is such subalgebra that if $a \in C(X)$ that for all $\varepsilon > 0$ number of such spaces H_g that $\|a|_{H_g}\| > \varepsilon$ is finite.

10.2 Noncommutative algebraic generalization of infinite covering

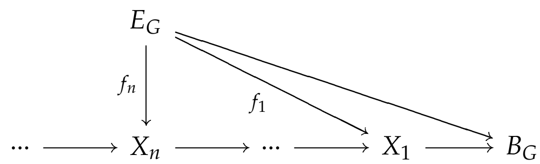
Let A be C^* algebra and

$$... \qquad A_n \longleftrightarrow ... \longleftarrow A_1 \longleftarrow A$$

sequence of finite coverings, and corresponding sequence of covering groups can be included into following diagram:



Also suppose that $\bigcap_{n \in \mathbb{N}} \ker(G \to G_n) = \{e\}$. Milnor's construction [30] provides such infinite covering space B_G that $\pi_1(B_G) \approx G$. Universal covering of this space is usually denoted by $E_G \to B_G$, G acts on E_G and $E_G/G \approx B_G$. This covering induces following diagram: This diagram induces following of coverings:



where $G(X_n, B_G) \approx G_n$, $\forall n \in \mathbb{N}$. Let $U \in E_G$ fundamental domain. For all elements $g_1, ..., g_m \in G_n$ we will select such representatives $\widetilde{g}_i \in G$ that set $\overline{\bigcup_{i=1,..,m} \widetilde{g}_i U_n}$ is connected. In this case $U_n = \bigcup_{g \in G_n} \widetilde{g} U$ is fundamental domain of $X_n \to B_G$ covering.

Let $A \to B(H)$ be GNS representation. Let $\widetilde{H} = \bigoplus_{g \in G} H_g$ is Hilbertian sum. Constructed algebra of infinite coverings subalgebra of $B(\widetilde{H})$. GNS representation is Hilbertian sum of irreducible representations. Irreducible representations of A will be indexed by set Λ i.e. $r_\lambda : A \to B(H_\lambda)$, $(\lambda \in \Lambda)$. Hilbert space of GNS representation is following Hilbert sum $H = \bigoplus_{\lambda \in \Lambda} H_\lambda$. According to [29] . Let $r : A_n \to B(H')$ any irreducible representation. According to [33] there exist irreducible limitation $r' : A \to B(K)$, where $K \subset H'$. A is hereditary subalgebra of A_n because A_n is finitely generated projective module. According to [33] $K = H'$ or it is such unique $\lambda \in Lambda$ that $H' = H_\lambda$ For any representation $r_\lambda : A \to H$ unique extension $r'_\lambda : A_n \to H$ will be fixed. Let $\phi_\lambda : A_n \to \mathbb{C}$ be positive functional which defines representation r'_λ. By $\phi_{\lambda g}$ denote following positive functional

$$\phi_{\lambda g}(a) = \phi_\lambda(ga);\ \forall a \in A_n, g \in G_n.$$

Representation defined by $\phi_{\lambda g}$ has same limitation on A as ϕ_λ one. By H_g denote Hilbertian sum of spaces of representations $\phi_{\lambda g}$, $\forall \lambda \in \Lambda$. So Hilbert space of GNS representation of A_n is Hilbertian is a direct sum $H_n = \bigoplus_{g \in G_n} H_g$. For all covering group $G_n = \{g_{n,1}, ..., g_{n,m}\}$ we will select such representatives $\widetilde{g}_{n,1}, ..., \widetilde{g}_{n,m} \in G$ that $U_n = \bigcup_{i=1,..,m} \widetilde{g}_{n,i} U$ is connected fundamental domain. Fundamental domains are selected by such way that if $i < j$ then $U_i \subset U_j$. Selection of these representatives enable us define action of A_n on Hilbertian sum $\widetilde{H} = \bigoplus_{g \in G} H_g$. Actions of algebras A_n are compatible with finite coverings $A_i \to A_j$. Let $B \in B(\widetilde{H})$ be norm completion of algebra generated by all elements $a \in A_n$, $n \in \mathbb{N}$. Let $\widetilde{A} \subset B$ be such subalgebra that for all $\varepsilon > 0$ number of elements $g \in G$ which satisfy condition $\|a|_{H_g}\| > \varepsilon$ is finite.

Definition 10.1. In this situation algebra $\widetilde{A}$ is named *generalization of infinite covering*.

11 Analogy with Kummer extensions

Here analogue with algebraic field extensions is considered. Let K be a field $\overline{K}$ (resp. K_{sep})is its algebraic (resp. separable) closure and $\mathfrak{G} = G(\overline{K}/K)$ is Galois group. Cyclic extension [**?**] L of K ($K \subset L \subset \overline{K}$) have such Galois group $\mathfrak{H} \subset \mathfrak{G}$ that $\mathfrak{G}/\mathfrak{H}$ is finite cyclic group. If number of elements of $\mathfrak{G}/\mathfrak{H}$ is equal to $n \in \mathbb{N}$ then $\mathfrak{G}/\mathfrak{H}$ is isomorphic to group of n-th roots of unity in $\mathbb{C}$. This isomorphism may be regarded as character χ of $\mathfrak{G}$ with the kernel $\mathfrak{H}$; such a character which if order of n, will be said *attached to L*. If α is representative in $\mathfrak{G}$ of generator of $\mathfrak{G}/\mathfrak{H}$, there is one and only one character χ of $\mathfrak{G}$ that $\chi(\alpha) = e^{2\pi i/n}$. Conversely if χ any homomorphism from $\mathfrak{G}$ into $\mathbb{C}^{\times}$; it is a character of order n; its kernel is open subgroup $\mathfrak{H} \in \mathfrak{G}$ with cyclic subgroup of order n and subfield $L \subset K_{sep}$ corresponding to $\mathfrak{H}$ is cyclic of degree n over K; we will than say that L is *attached to L*. If K contains distinct n roots of 1; then these make up a cyclic group E of order n, if K is of characteristic $p > 1$, or assumption implies that n is prime to p. Let ψ be an isomorphism of E onto group of n-th roots of 1 in $\mathbb{C}$; this will be determined uniquely if we choose a generator ϵ_1 of E and prescribe $\psi(\epsilon_1) = e^{2\pi i/n}$. Take any $\xi \in K^{\times}_{sep}$, and let x be any one of roots of equation $X^n = \xi$ in $\overline{K}$; when $x \in K^{\times}_{sep}$, and equation $X^n = \xi$ has n distinct roots ϵx with $\epsilon \in E$. In particular, for each $\sigma \in \mathfrak{G}$ x^{σ} must be one of the roots, so that $x^{\sigma}x^{-1}$ is in E. Now put

$$\chi_{n,\xi}(\sigma) = \psi(x^{\sigma}x^{-1}); \qquad (52)$$

as $E \in K$, the right-hand side does not change if we replace x by ϵx with $\epsilon \in E$ and is therefore independent of choice of a root x for $X^n = \xi$. For similar reason, we have, for all, $\rho, \sigma \in \mathfrak{G}$;

$$x^{\rho\sigma}x^{-1} = (x^{\rho}x^{-1})^{\sigma}(x^{\sigma}x^{-1}) = (x^{\rho}x^{-1})(x^{\sigma}x^{-1}),$$

and therefore

$$\chi_{n,\xi}(\rho\sigma) = \chi_{n,\xi}(\rho)\chi_{n,\xi}(\sigma),$$

and therefore shows that $\chi_{n,\xi}$ is a character on $\mathfrak{G}$. Take now any $\eta \in K^{\times}$, and call y a root of $X^n = \eta$; then xy is root of $X^n = \xi\eta$, and we have for all $\sigma \in \mathfrak{G}$):

$$(xy)^{\sigma}(xy)^{-1} = (x^{\sigma}x^{-1})(y^{\sigma}y^{-1})$$

end therefore

$$\chi_{n,\zeta\eta} = \chi_{n,\zeta}\chi_{n,\eta},$$

which shows that $\xi \mapsto \chi_{n,\xi}$ is a morphism of $K^{\times}$ into group of characters of $\mathfrak{G}$. It is obvious that $\chi_{n,\xi}$ is trivial if $X^n = \xi$ has one root, hence all its roots, in K, i.e. if $\xi \in (K^{\times})^n$; in other words, $(K^{\times})^n$ is kernel of $\xi \mapsto \chi_{n,\xi}$. It would be easy to show that the image of $K^{\times}$ under that morphism consists of all the characters of $\mathfrak{G}$ whose order divides n, but this will not be needed. Let us generalize this construction. Let A be C^* - algebra, $U(A)$ group of its unitary elements, $U_0(A) \in U(A)$ subgroup homotopic to unity elements, $[U(A)] = U(A)/U_0(A)$ factorgroup. By $[[U(A)], [U(A)]]$ denote commutator of $[U(A)]$, by $[U(A)]_{ab}$ denote factorgroup $[U(A)]/[[U(A)], [U(A)]]$. Let $\mathrm{Tors}([U(A)]_{ab}) \in [U(A)]_{ab}$ be subgroup of elements which have finite period. Factorgroup $[U(A)]_{ab\ free} = [U(A)]_{ab}/\mathrm{Tors}([U(A)]_{ab})$ is free Abelian group. Let $u_1, ..., u_p \in U(A)$ such unitary elements that:

1. u_i satisfy conditions of definition 6.9 for $i = 1, ..., p$.

2. Classes $\overline{u}_i \in [U(A)]_{ab\ free}$ are linearly independent.

According to example **??** and/or example **??** $\forall \in \mathbb{N}$ one can Abelian construct finite covering $A \to B$ that there are such unitary elements $v_1, ..., v_k \in B$ that $v_k^n = u_k$. By $\mathfrak{G}$ denote Abelian Galois group of this covering. This group can be represented as following direct sum.

$$\mathfrak{G} = \bigoplus_{j=1,...,k} \mathfrak{G}_j; \; \mathfrak{G}_j \approx \mathbb{Z}_n \forall j \; (1 \leq j \leq k);$$

And summand $\mathfrak{G}_j$ is generated by automorphism $\sigma \in \mathfrak{G}$ which acts on B by following way:

$$v_j \mapsto e^{2\pi i/n} v_j;$$

$$v_l \mapsto v_l; \; l \neq j.$$

Now we can define character $\xi_{n,\overline{u}_j}$ on defined as

$$\xi_{n,\overline{u}_j}(\sigma) = \sigma(v_j) v_j^{-1};$$

This equation can be regarded as analogue of equation 52.

12 Generalization of infinite covering

According to section 1 C^* it is following mapping:

TOPOLOGY	ALGEBRA
Locally compact space	C^* - algebra
Compact space	Unital C^* - algebra
Continuous map	*-homomorpfism

This map excludes generalization of infinitely listed coverings by following reasons. Let X be compact Hausdorff space and $p : Y \to X$ be infinitely listed covering, then Y is not compact, $C_0(X)$ is unital, but $C_0(Y)$ is not unital. Homomorphism $C(p) : C_0(X) \to C_0(Y)$ which correspond to p does not exist.So one should generalize notion of *- homomorphism for generalization of infinitely listed coverings.

12.1 Hurewicz homomorphism with respect to covering

Hurewicz homomorphism is in general homomorphism from noncommutative group to commutative one. So it can be decomposed by following way:

$$\pi_1(X) \to \pi_{ab}(X) \to H_1(X),$$

where $\pi_{ab}(X)$ is Abelian group defined as $\pi_{ab}(X) = \pi_1(X)/[\pi_1(X), \pi_1(X)]$
Algebraic topology has good notion of fundamental group. However good noncommutative generalization of fundamental group is not known. But every covering $\widetilde{X} \to X$ defines covering group $G(\widetilde{X}, X)$ which is factorgroup of fundamental group. If this group has natural structure of subgroup then one can define natural homomorphism $G(\widetilde{X}, X) \to H_1(X)$.

Since $H_1(X)$ is Abelian we can take into account Abelian covering projections only (see section 6) i.e. coverings with Abelian covering transformation group. Abelian group is simultaneously subgroup and factorgroup if is direct summand. So if $G(\widetilde{X}, X)$ is direct summand of $\pi_{ab}(X)$ then it is natural homomorphism $G(\widetilde{X}, X) \to H_1(X)$.

Definition 12.1. Let $\pi : \widetilde{X} \to X$ be Abelian covering and $G(\widetilde{X}, X)$ is direct summand of $\pi_{ab}(X)$. Natural homomorphism $G(\widetilde{X}, X) \to H_1(X)$ is a *Hurewicz homomorphism with respect to* π.

Let us generalize this definition. Fundamental group is not defined for noncommutative C^* - algebras. However if $G(\widetilde{X}, X)$ is direct summand of $\pi_{ab}(X)$ is also direct summand for all intermediate subgroup G i.e. $G(\widetilde{X}, X) \subset G \subset \pi_{ab}(X)$.
This observation enable us define generalization of Hurewicz homomorphism with respect to covering by following way.

Definition 12.2. Let $\pi : A \to B$ be such Abelian covering of C^* - algebras that for all Abelian coverings $B \to C$ group $G(B, A)$ is direct summand of $G(C, A)$.*Hurewicz homomorphism with respect to* π $A \to B$ is natural homomorphism from $G(B, A)$ to $K^1(A)$.

Let us generalize this definition. Fundamental group is not defined for noncommutative C^* - algebras. However if $G(\widetilde{X}, X)$ is direct summand of $\pi_{ab}(X)$ is also direct summand for all intermediate subgroup G i.e. $G(\widetilde{X}, X) \subset G \subset \pi_{ab}(X)$.
This observation enable us define generalization of Hurewicz homomorphism with respect to covering by following way.

12.2 Noncommutative Hurewicz homomorphism

Noncommutative generalization of Hurewicz homomorphism is not group homomorphism. It is a set of homomorphism's conditions. In particular cases this conditions define unique group homomorphism. In general this homomorphism does not exist and is not unique. Let G, H be finitely generated Abelian groups and $f : G \to H$ is group homomorphism. Let G_{tors} (resp. H_{tors} be torsion of G (resp. H). Then it is following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G_{tors} & \longrightarrow & G & \longrightarrow & G/G_{tors} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle f_{tors}} & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle \overline{f}} & & \\
0 & \longrightarrow & H_{tors} & \longrightarrow & H & \longrightarrow & H/H_{tors} & \longrightarrow & 0
\end{array}$$

Homomorphism f uniquely defines both f_{tors} and $\overline{f}$, but not vice versa. So f_{tors} and $\overline{f}$ can be regarded as properties of f. If one of following conditions is satisfied

1. $G_{tors} \approx \{0\}$;
2. $G/G_{tors} \approx \{0\}$

3. $H_{tors} \approx \{0\}$

4. $H/H_{tors} \approx \{0\}$

then f_{tors} and $\overline{f}$ uniquely define f. Otherwise if A is C^* - algebra then it is following sequence:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ext}^1_{\mathbb{Z}}(K_0(A), K_0(\mathbb{C})) & \longrightarrow & KK^1(A, \mathbb{C}) & \longrightarrow & \mathrm{Hom}(K^1(A), K_0(\mathbb{C})) & \longrightarrow & 0 \\
 & & \downarrow \approx & & \downarrow \approx & & \downarrow \approx & & \\
0 & \longrightarrow & \mathrm{Rep}(K_0(A)_{tors}) & \longrightarrow & K^1(A) & \longrightarrow & \mathrm{Hom}(K^1(A), \mathbb{Z}) & \longrightarrow & 0
\end{array}$$

Construction of Hurewicz homomorphism generalization has properties of following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G_{tors} & \longrightarrow & G & \longrightarrow & \mathrm{Hom}(G/G_{tors}, \mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow f_{tors} & & \downarrow f & & \downarrow \overline{f} & & \\
0 & \longrightarrow & \mathrm{Rep}(K^0(A)_{tors}) & \longrightarrow & K^1(A) & \longrightarrow & \mathrm{Hom}(K^1(A), \mathbb{Z}) & \longrightarrow & 0
\end{array}$$

where $G = G(B|A)$ is Abelian covering group and $\mathrm{Rep}(G)$ means representation group $\forall G$ (G is finite Abelian group) .
Rather we would like construct properties of f_{tors} and $\overline{f}$. First of all note that $K^*(A) \approx KK^*_G(\mathbb{C}, A)$. There are canonical pairings

$$G_{tors} \times \mathrm{Rep}(G_{tors}) \to \mathfrak{A},$$

$$\mathrm{Rep}(K^0(A)_{tors}) \times K^0(A_{tors}) \to \mathfrak{A},$$

where $\mathfrak{A}$ is finite Abelian group.
So following pairing

$$\mathrm{Rep}(G_{tors}) \times K^0(A)_{tors} \to \mathfrak{A} \qquad (53)$$

can be regarded as analog of isomorphism $G_{tors} \approx \mathrm{Rep}(K^0(A)_{tors})$. Kasparov intersection product $KK^*_G(\mathbb{C}, \mathbb{C}) \otimes KK^*_G(\mathbb{C}, B) \to KK^*_G(\mathbb{C}, B)$. From

$$KK^*_G(\mathbb{C}, \mathbb{C}) \approx \mathrm{Rep}(G_{tors}),$$

$$KK^0_G(\mathbb{C}, B) \approx KK^0(\mathbb{C}, A) \approx K^0(A).$$

it follows that it is following pairing.

$$\mathrm{Rep}(G_{tors}) \times K^0(A) \to K^0(A).$$

Since $\mathrm{Rep}(G_{tors})$ is finite above pairing does not depend on infinite part of $K^0(A)$ i.e.

$$\mathrm{Rep}(G_{tors}) \times K^0(A)_{tors} \to K^0(A)_{tors}.$$

Above formula is in fact pairing (53).
There are following natural pairings.

$$G/G_{tors} \times \mathrm{Hom}(G/G_{tors}, \mathbb{Z}) \to \mathbb{Z}.$$

$$K^1(A) \times \mathrm{Hom}(K^1(A), \mathbb{Z}) \to \mathbb{Z}.$$

So following pairing

$$K^1(A) \times \mathrm{Hom}(K^1(A), \mathbb{Z}) \to \mathfrak{A}.$$

can be regarded as analogue of isomorphism $G/G_{tors} \approx K^0(A)$ From Kasparov intersection product it follows next pairing

$$KK^1_G(\mathbb{C}, \mathbb{C}) \times KK^1_G(\mathbb{C}, B) \to KK^0_G(\mathbb{C}, B) \tag{54}$$

If G is finitely generated Abelian group then

$$KK^1_G(\mathbb{C}, \mathbb{C}) \approx \mathrm{Hom}(G, \mathbb{Z}) \approx \mathrm{Hom}(G/G_{tors}, \mathbb{Z});$$

$$KK^1_G(B, \mathbb{C}) \approx KK^1(A, \mathbb{C}) \approx K^1(A).$$

So pairing (54) can be regarded as isomorphism

12.3 Construction Hurewicz homomorphism generalization

Now we have all ingredients for construction of Hurewicz homomorphism generalization. Let (A, h) be noncommutative generalization of pointed space (see definition **??**), $\pi : A \to B$ be Abelian covering which satisfies conditions of definition 12.2, and covering group $G = G(B, A)$ is finitely generated (Abelian) group.
Construction of Hurewicz homomorphism generalization with respect to π includes following steps.

1. It is natural isomorphism: $K^*_G(B) \to K^*(A)$;
2. It is natural isomorphism $G \sim KK^1_G(\mathbb{C}, \mathbb{C})$;
3. $KK^1_G(\mathbb{C}, \mathbb{C})$ acts on $KK^0_G(B, \mathbb{C}) \sim K^0_G(B) \sim K^0(A)$, So G acts on $K^0(A)$, it is pairing $G \times K^0(A) \to K^1(A)$;
4. Hurewicz homomorphism generalization with respect to π is defined as

$$G \ni g \mapsto (gh - h) \in K^1(A). \tag{55}$$

Definition 12.3. Let (A, h) be noncommutative generalization of pointed space and $\pi : A \to B$ be Abelian covering which satisfies conditions of definition 12.2, and covering group $G = G(B, A)$ is finitely generated (Abelian) group. An Abelian group homomorphism $\phi : G \to K^1(A)$ is called *Hurewicz homomorphism generalization with respect to* π if ϕ is defined by equation (55).

Remark 12.4. *Functionality of Hurewicz homomorphism*. Let $f : A \to B$ be *- homomorphism, $\overline{f}$ is a homomorphism of fundamental groups with respect to $A \to \widetilde{A}$, $B \to \widetilde{B}$ coverings, (B, h), $(A, K^1(f)(h))$ generalizations of pointed spaces. Then following natural diagram

$$\begin{array}{ccc} G(\widetilde{B}|B) & \xrightarrow{\overline{f}} & G(\widetilde{A}|A) \\ \downarrow & & \downarrow \\ K^1(B) & \xrightarrow{K^1(f)} & K^1(B) \end{array}$$

is commutative. Vertical arrows of above diagram are generalizations of Hurewicz homomorphism defined by pairs (B, h), $(A, K^1(f)(h))$.

Let us remind universal coefficient theorem of KK theory

Theorem 12.5. *[41] Let A and B be separable C^* algebras with $A \in N$. Then there is a short exact sequence*

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B)) \xrightarrow{\delta} KK^*(A, B) \xrightarrow{\gamma} \mathrm{Hom}(K_*(A), K_*(B)) \to 0. \tag{56}$$

The map γ has degree 0 and δ has degree 1. The sequence is natural and splits unnaturally. So if $K_*(A)$ is divisible or $K_*(B)$ is divisible, then γ is isomorphism.

Particular case of theorem56 is following sequence:

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(K_0(A), \mathbb{Z}) \xrightarrow{\delta} K^1(A) \xrightarrow{\gamma} \mathrm{Hom}(K_1(A), \mathbb{Z}) \to 0.$$

Similarly if G is finitely generated Abelian group then it is following exact sequence:

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(KK^0_G(\mathbb{C}, \mathbb{C}), \mathbb{Z}) \xrightarrow{\alpha} KK^1_G(\mathbb{C}, \mathbb{C}) \xrightarrow{\beta} \mathrm{Hom}(KK^1_G(\mathbb{C}, \mathbb{C}), \mathbb{Z}) \to 0.$$

Generalization of Hurewicz homomorphism induces following natural homomorphisms between above exact sequences.

$$\begin{array}{ccccc} \mathrm{Ext}^1_{\mathbb{Z}}(KK^0_G(\mathbb{C}, \mathbb{C}), \mathbb{Z})) & \xrightarrow{\alpha} & KK^1_G(\mathbb{C}, \mathbb{C}) & \xrightarrow{\beta} & \mathrm{Hom}(KK^1_G(\mathbb{C}, \mathbb{C}), \mathbb{Z}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Ext}^1_{\mathbb{Z}}(K_0(A), \mathbb{Z}) & \xrightarrow{\alpha} & K^1(A) & \xrightarrow{\beta} & \mathrm{Hom}(K_1(A), \mathbb{Z}) \end{array}$$

13 Appendix A. Suspension and loop algebras

13.1 Loop space and suspension in topology

Definition 13.1. [34] Let $p : \tilde{X} \to X$ covering projection. It is clear that there is a *group of self-equivalences* of p (a self-equivalence is a homeomorphism $f : \tilde{X} \to \tilde{X}$ such that $p \circ f = p$). We denote this group by $G(\tilde{X}|X)$. This group is also called the *group of covering transformations* of p.

13.2. *Fuction spaces.* If X and Y are topological spaces, we let Y^X denote the set of all continuous functions functions $f : X \to Y$. We give this set a topology, called the *compact-open topology*, by taking as a subbase for the topology all sets of the form $N_{K,U} = \{f : f(K) \subset U\}$, $K \subset U$ compact, $U \subset X$ open.

13.3. *Base points.* Algebraic topology often have to consider not just topological space X but rather a space X together with distinguished point $x_0 \in X$ called the *base point*. The pair (X, x_0) is called a *pointed space* (one also speaks of pointed sets). When we are concerned with pointed spaces (X, x_0), (Y, y_0), etc. we always require that all functions $f : X \to Y$ shall preserve base point, i.e. $f(x_0) = y_0$, and that all homotopies $F : X \times I \to Y$ be relative to base point, i.e. $F(x_0, t) = y_0$, $\forall t \in I$ unless an explicit disclaimer to be contrary is made. We shall use the notation $[X, x_0, Y, y_0]$ to denote the homotopy classes of base point preserving functions, where homotopies are relx_0, of course. $[X, x_0, Y, y_0]$ is a pointed space with base point f_0 the constant function: $f_0(x) = y_0$, $\forall x \in X$. If (X, x_0), (Y, y_0) are pointed spaces then we have the space $(Y, y_0)^{(X,x_0)}$ of base point preserving functions. We use the notation $X \vee Y$ for the subspace $Z \times \{x_0\} \cup \{z_0\}X$. It can be thought of as the result of taking the disjoint union $Z \cup X$ and identifying z_0 with x_0, $Z \vee X$ is again a pointed space with base point (z_0, x_0). Given maps $f : (X, x_0) \to (X', x'_0)$ and $g : (Y, y_0) \to (Y', y'_0)$ the map $f \times g : X \times Y \to X' \times Y'$ maps $X \vee Y$ into $X' \vee Y'$ and so induces a map $f \wedge g : X \wedge Y \to X' \wedge Y'$.

13.4. If both (X, x_0), Y, y_0 are pointed spaces then we define the *smash product* $(X \wedge Y, *)$ to be the quotinent

$$X \wedge Y = X \times Y / X \vee Y$$

with the point $* = p(X \vee Y)$ as base point. Here $p : X \times Y \to X \vee Y$ is the projection. For any $(x, y) \in X \times Y$ we denote $p(x, y) \in X \wedge Y$ by $[x, y]$.

Theorem 13.5. *(See [2]) If $(X, x_0), (Y, y_0), (Z, z_0)$ are pointed spaces, X, Z Hausdorff and Z locally compact, then there is a natural equivalence*

$$A : [Z \wedge X, *; Y, y_0] \to [X, x_0, (Y, y_0)^{(Z,z_0)}, f_0] \tag{57}$$

defined by $A[f] = [\hat{f}]$, where if $f : Z \wedge X \to Y$ is a map then $\hat{f} : Z \wedge X \to Y$ is a map given by $(\hat{f}(x)(z) = f[z, x]$.

13.6. If (Y, y_0) is a pointed space, we define the *loop space* $(\Omega Y, \omega_0)$ of Y to be function space

$$\Omega Y = (Y, y_0)^{(S^1, s_0)}$$

with constant loop ω_0 $(\omega_0(s)) = y_0$, $\forall s \in S^1)$ as base point.

13.7. If (X, x_0) we define *suspension* $(SX, *)$ to be the smash product $(S^1 \wedge X, *)$ of X with the 1-sphere.

Corollary 13.8. *(See [2]) If both $(X, x_0), (Y, y_0)$ are pointed spaces and X is Hausdorff, then there is a natural equivalence*

$$A : [SX, *; Y, y_0] \to [X, x_0, \Omega Y, y_0]. \tag{58}$$

Definition 13.9. A *group* is a pointed set (G, e) with *multiplication* $\mu : G \times G \to G$ and an *inverse* $\nu : G \to G$ such that the following diagrams commute:

1. $$\begin{array}{ccccc} G & \xrightarrow{(e,1)} & G \times G & \xleftarrow{(1,e)} & G \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle 1} & \\ & & G & & \end{array}$$

   (e is two sided identity)

2. $$\begin{array}{ccc} G \times G \times G & \xrightarrow{\mu \times 1} & G \times G \\ {\scriptstyle 1 \times \mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ G \times G & \xrightarrow{\mu} & G \end{array}$$

   (associativity)

3. $$\begin{array}{ccccc} G & \xrightarrow{(\nu,1)} & G \times G & \xleftarrow{(1,\nu)} & G \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle 1} & \\ & & G & & \end{array}$$

   (inverse)

Here $e : G \to G$ is the constant map $e(g) = e$, $\forall g \in G$. $(e, 1)$ means the map such that $(e, 1)(g) = (e, g)$, etc. G is called *commutative* or *abelian* if in addition the following diagram commutes

$$\begin{array}{ccc} G \times G & \xrightarrow{T} & G \times G \\ & {\scriptstyle 1}\searrow \qquad \swarrow{\scriptstyle 1} & \\ & G & \end{array}$$

where $T : G \times G \to G \times G$ is the *switch* map $T(g_1, g_2) = (g_2, g_1)$, $\forall (g_1, g_2) \in G \times G$.

Motivated by this definition of a group we make the following definitions.

Definition 13.10. An H - *space* is a pointed space (K, k_0) with a multiplication map $\mu K \times K \to K$ such that k_0 is a *homotopy identity*, that is, the diagram:

1. $$\begin{array}{ccccc} K & \xrightarrow{(k_0,1)} & K \times K & \xleftarrow{(1,k_0)} & K \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle 1} & \\ & & K & & \end{array}$$

   commutes up to homotopy: $\mu \circ (1, k_0) \simeq 1 \simeq \mu \circ (k_0, 1)$. We say μ

2. We say μ is *homotopy associative* if the diagram

   $$\begin{array}{ccc} K \times K \times K & \xrightarrow{\mu \times 1} & K \times K \\ {\scriptstyle 1\times\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ K \times K & \xrightarrow{\mu} & K \end{array}$$

   commutes up to homotopy: $\mu \circ (\mu \times 1) \simeq \mu \circ (1 \times \mu)$

3. A map $\nu(K, k_0) \to (K, k_0)$ is called a *homotopy inverse* if the diagram

   $$\begin{array}{ccccc} K & \xrightarrow{(\nu,1)} & K \times K & \xleftarrow{(1,\nu)} & K \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle 1} & \\ & & K & & \end{array}$$

   commutes up to homotopy: $\mu \circ (\nu, 1) \simeq k_0 \simeq \mu \circ (1, \nu)$

4. We say μ is *homotopy commutative* if the diagram

5. $$\begin{array}{ccc} K \times K & \xrightarrow{T} & K \times K \\ & {\scriptstyle 1}\searrow \qquad \swarrow{\scriptstyle 1} & \\ & K & \end{array}$$

   commutes up to homotopy: $\mu \circ T \simeq \mu$.

An H - *group* is an H space with (K, k_0) with homotopy associative multiplication μ and homotopy inverse ν.
Note: All maps and homotopies should be relative to the base point $K \times K$ has base point (k_0, k_0).

Proposition 13.11. *(See [2]) If (K, k_0) is an H - group with multiplication μ and homotopy inverse ν, then for every pointed space the set*

$$[X, x_0; K, k_0]$$

can be given the structure of a grpup if we define the product $[f] \cdot [g]$ to be the homotopy class of the composition

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} K \times K \xrightarrow{\mu} K.$$

Here Δ is the diagonal map given by $\Delta(x) = (x, x)$. The identity of the group is the class $[k_0]$ of the constant map, and inverse is given by $[f]^{-1} = [\nu \circ f]$. If μ is homotopy commutative, then $[X, x_0; K, k_0]$ is abelian. Every map $f : (X, x_0) \to (Y, y_0)$ induces a homomorphism $f^* : [Y, y_0; K, k_0] \to [X, x_0 : K, k_0]$.

13.12. The most important example of H-group is the loop space $(\Omega Y, \omega_0)$ for any pointed space (Y, y_0). Before defining

$$\mu : \Omega Y \times \Omega Y \to \Omega Y$$

we make the following remark: there is an obvious homeomorphism

$$(I/\{0,1\}, *) \to (S^1, *),$$

and in fact we have a homeomorphism

$$(Y, y_0)^{(S^1, *)} \simeq (Y, y_0)^{(I, \{0,1\})}$$

For many purposes it will be more convenient to regard ΩY as being $(Y, y_0)^{(I, \{0,1\})}$, i.e. as the space of paths beginning end ending at y_0. It these terms we define μ by

$$\mu(\omega, \omega')(t) = \begin{cases} \omega(2t) & 0 \le t \le 1/2, \\ \omega'(2t-1) & 1/2 \le t \le 1. \end{cases}$$

What we have done, in fact, is define a map

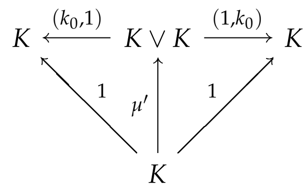
$$\overline{\mu} : \Omega Y \times I \to Y$$

which is continuous, Then the exponential law gives our

$$\mu : \Omega Y \times \Omega Y \to \Omega Y.$$

Definition 13.13. An H - *cogroup* is a pointed space with continuous *comultiplication* $\mu' : K \to K \vee K$ such that k_0 is a *homotopy identity*, i.e. the diagram

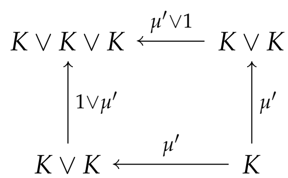
1.



commutes up to homotopy (here $(k_0, 1)$ denotes the map such that $(k_0, 1)(k, k_0) = k_0$ and $(k_0, 1)(k_0, k) = k \ \forall k \in K$).

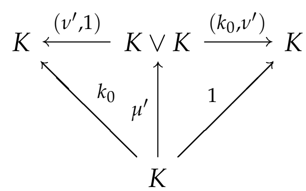
2. μ' is further required to be *homotopy associative*, i.e. the diagram

3.



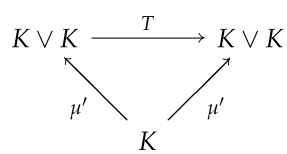
must commute up to homotopy.

4. Moreover K must have *homotopy inverse* $\nu' : K \to K$ such that the diagram



commutes up to homotopy.

5. Finally K is called a *homotopy commutative H*-cogroup if in addition the diagram



commutes up to homotopy.

We now get the proposition dual to 13.11.

Proposition 13.14. *(See [2]) If (K, k_0) is an H-cogroup with comultiplication μ' and homotopy inverse ν', then for every pointed space (X, x_0) the set*

$$[K, k_0; X, x_0] \tag{59}$$

can be given the structure of a group if we define the product $[f] \cdot [g]$ to be a homotopy class of the composition

$$K \xrightarrow{\mu'} K \vee K \xrightarrow{f \vee g} X \vee X \xrightarrow{\Delta'} X.$$

Here Δ' is folding map given by $\Delta'(x, x_0) = x = \Delta'(x_0, x)$. The identity of the group is the class $[x_0]$ of constant map, and the inverse is given by $[f]^{-1} = [f \circ \nu']$. If map $f : (X, x_0) \to (Y, y_0)$ induces a homomorphism

$$f_* : [K, k_0; X, x_0] \to [K, k_0; Y, y_0].$$

13.15. If the principal example of an H-group is ΩY, any Y, then in the ligtht of 13.8 it should be no surprise that the principal example of an H-cogroup is SX, any X. Before defining

$$\mu' : SX \to SX \vee SX$$

we make following remark: because of the homeomorphism $(S^1, s_0) \simeq (I/\{0,1\}, *)$ we may regard SX as being the quotionent

$$I \times X / \{0\} \times X \cup I \times \{x_0\} \cup \{1\} \times X. \tag{59}$$

If $p : I \times X \to SX$ is the projection, then we shall denote $p(x, t) \in SX$ by $[t, x]$, $t \in I$, $x \in X$. In these terms we define μ' as follows:

$$\mu'[t, x] = \begin{cases} ([2t, x], x_0), & 0 \le t \le 1/2 \\ (x_0, [2t - 1, x]) & 1/2 \le t \le 1. \end{cases}$$

The homotopy inverse $\nu' : SX \to SX$ is defined as follows:

$$\nu'[t, x] = [1 - t, x], \ t \in I, \ x \in X.$$

Proposition 13.16. *(See [2]) The adjoint correspondence*

$$A : [SX, *; Y, y_0] \to [X, x_0, \Omega Y, \omega_0]$$

is an isomorphism of groups.

13.2 Noncommutative Suspension and Loop Space

13.2.1 Generators and relations

We begin with the construction of the C^*-algebra on a properly chosen set of generators and relations. Our treatment is based that of [3], but includes changes to make it more suitable for our purposes. If G is any set, we denote by $F(G)$ the free associative complex *-algebra (without identity) on the set G. Thus, $F(G)$ consists of all polynomials in the noncommuting variables $C \coprod G^*$ (disjoint union), with complex coefficients and no constant term. By definition, any function p from G to a C^*-algebra A extends to a unique *-homomorphism, which we also call p, from $F(G)$ to A.

A set R of relations on G is a collection of statements about the elements of G which make sense for elements of a C^* -algebra. Possible relations include statements of the form "$\|x\| \in S$"', where $x \in F(G)$ and $S \in \mathbb{R}$, "x is positive," or the statement that some equation in the variables $G \coprod G^*$ and some unknowns has a solution, or that some function from a topological space into G is continuous. Note that Blackadar considers only relations of the form $\|x\| \leq \eta$ for $\eta > 0$ and x in the unitization $F(G)^+$ of $F(G)$. (We do not allow relations involving elements of $F(G)^+ - F(G)$ because they do not make sense in a nonunital C^*- algebra. However, it is perfectly possible for R to include the relations $eg = geg$ for some fixed $e \in G$ and all $g \in G \coprod G^*$.)

Definition 13.17. Let (G, R) be a set of generators and relations. (That is, G is a set and R is a set of relations on G.) Then a representation of (G, R) in a C^*-algebra A is a function $\rho : G \to A$ such that the elements $\rho(g)$ for $g \in G$ satisfy the relations R in A. A representation on a Hilbert space H is a representation in $L(H)$.

Definition 13.18. (Compare [3]). A set (G, R) of generators and relations is admissible if the following conditions hold:

1. The function from G to the zero C^*-algebra is a representation of (G, R).
2. If ρ is a representation of (G, R) in a C^*-algebra A, and if B is a C^*-subalgebra of A which contains $\rho(G)$, then ρ is a representation of (G, R) in B.
3. If ρ is a representation of (G, R) in a C^*-algebra A, and if $\psi : A \to B$ is a surjective homomorphism, then $\psi \circ \rho$ is a representation of (G, R) in B.
4. For every $g \in G$ there is a constant $M(g)$ such that $\|\rho(g)\| < M(g)$ for all representations p of (G, R).
5. If $\{\rho_\alpha\}$ is a family of representations of (G, R) on Hilbert spaces H_α then $g \mapsto \rho(g) = \oplus_\alpha \rho_\alpha(g)$ is a representation of (G, R) on $H = \oplus H_\alpha$. (That is, the elements $\rho(g)$, which are in $B(H)$ by (4), in fact satisfy the relations R.)

Note that, in the presence of (3), condition (1) is equivalent to "there exists a representation of (G, R)." Also note that, for relations of the sort considered by Blackadar, (2) and (3) are automatic and (5) follows from (4). The universal C^*-algebra on the generators G and relations R is a C^*- algebra $C^*(G, R)$ with a representation ρ of (G, R) in $C^*(G, R)$, such that, given any representation σ of (G, R) in a C^*-algebra B, then is a unique homomorphism $\psi : C^*(G, R) \to B$ such that $\sigma = \psi \circ \rho$. If (G, R) is admissible, then $C^*(G, R)$ exists and, following Blackadar, can be obtained as the Hausdorff completion of $F(G)$ in the C^* -seminorm $\|x\| = \sup|\rho(x)\|$ where ρ is a representation of (G, R). Note that condition (4) guarantees that $\|x\| < \infty$ for $x \in G$, and that condition (5) guarantees that the obvious map from G to $C^*(G, R)$ is in fact a representation. Admissibility, or something close to it, is also necessary for the existence of $C^*(G, R)$. Condition (1) is needed, since otherwise there may be no representations at all; conditions (2) and (3) are needed to ensure that the notion of a universal C^*-algebra is sensible, and without conditions (4) and (5) it will not be possible to construct a universal C^* -algebra. However, if the relations R also make sense in a pro-C^* -algebra, then a pro-C^*-algebra with the required properties will exist under much weaker conditions, as in the following definition. A representation of (G, R) in a pro-C^*-algebra has the obvious meaning.

Definition 13.19. A set (G, R) of generators and relations is called weakly admissible if the following conditions are satisfied:

1. The function from G to the zero C^*-algebra is a representation of (G, R).

2. If ρ is a representation of (G, R) in a C^*-algebra A, and if B is a C^*-subalgebra of A which contains $\rho(G)$, then ρ is a representation of (G, R) in B.

3. If ρ is a representation of (G, R) in a C^*-algebra A, and if $\varphi : A \to B$ is a surjective homomorphism, then $\varphi \circ \rho$ is a representation of (G, R) in B.

4. For every $g \in G$ there is a constant $M(g)$ such that $\|\rho(g)\| < M(g)$ for all representations p of (G, R).

5. If $\{\rho_\alpha\}$ is a family of representations of (G, R) on Hilbert spaces H_α then $g \mapsto \rho(g) = \oplus_\alpha \rho\alpha(g)$ is a representation of (G, R) on $H = \oplus H_\alpha$. (That is, the elements $\rho(g)$, which are in $B(H)$ by (4), in fact satisfy the relations R.)

6. If A is a pro-C^*-algebra, and $\rho : G \to A$ is a function such that, for every $p \in S(A)$, the composition of ρ with $A \to A_p$ is a representation of (G, R) in A_p, then ρ is a representation of (G, R).

7. If $\rho_1, ..., \rho_n$ are representations of (G, R) in C^*-algebras $A_1, ..., A_n$ then $g \mapsto (\rho_1(g), ..., \rho_n(g))$ is a representation of (G, R) in $A_1 \oplus ... \oplus A_n$.

Proposition 13.20. *[1] Let (G, R) be a weakly admissible set of generators and relations. Then there exists a pro-C^* -algebra C*(G,R), equipped with a representation $\rho : G \to C^*(G, R)$ of (G, R), such that, for any representation σ of (G, R) in a pro-C^*-algebra B, there is a unique homomorphism $\varphi : C^*(G, R) \to B$ satisfying $\sigma = \varphi \circ \rho$. If (G, R) is admissible, then $C^*(G, R)$ is a C^*-algebra.*

We would like to generalize notion of a loop space.

13.2.2 The Dual Group

Definition 13.21. A *dual group in the category of unital C^*-algebras* is a quadruple (A, μ, ι, χ) consisting of a unital C^*-algebra A and unital homomorphisms $\phi : A \to A *_{\mathbb{C}} A$, $\iota : A \to A$, and $\chi : A \to \mathbb{C}$, satisfying certain conditions. To state them, we introduce the unital homomorphisms $\varepsilon_A : \mathbb{C} \to A$, given by $\varepsilon_A(\lambda) = \lambda \cdot 1$, and $\delta_A : A *_{\mathbb{C}} A \to A$, mapping each copy of A in the free product identically onto A. Then the conditions are:

1. $(\mu, *\mathrm{id}_A) \circ \mu = (\mathrm{id}_A * \mu) \circ \mu$, (as maps from A to $A *_{\mathbb{C}} A *_{\mathbb{C}} A$).
2. $\iota^2 = \mathrm{id}_A$.

3. $\delta_A \circ (\iota * \mathrm{id}_A) \circ \mu = \delta_A \circ (\mathrm{id}_A * \iota) \circ \mu = e_A \circ \chi$.
4. $(\chi * \mathrm{id}_A) \circ \mu = (\mathrm{id}_A * \chi) \circ \mu = \mathrm{id}_A$, where $\mathbb{C} *_{\mathbb{C}} A$ and $A *_{\mathbb{C}} \mathbb{C}A$ are identified in the obvious way with A.

To understand this definition, imagine that it is taking place in the category of sets, with the directions of all maps reversed, and with all category concepts replaced by their duals. This last operation means here that the free product $A *_{\mathbb{C}} A$ must be replaced by the product, and that the algebra $\mathbb{C}$ (which has a unique unital homomorphism ε_A to any unital C^*-algebra A) must be replaced by a one point set $\{e\}$ (which has a unique map from every other set). Rearranged in this manner, this definition simply says that A is a group, with multiplication $\mu : A *_{\mathbb{C}} A$, inversion $\iota : A \to A$, and identity $\chi(e) \in A$. For this reason, the object we have defined is also called a "group object in the opposite category to the category of unital C^*-algebras".

13.2.3 The Noncommutative Loop Space

Definition 13.22. If (A, α) is a pointed pro-C^*-algebra, than its suspension is the pro-C^*-algebra $\Sigma A = \{f : S^1 \to A \text{ continuous } : \alpha(f(\zeta)) \cdot 1 = f(1), \ \forall \zeta \in S^1\}$, together with the homomorphism $ev_1 : \Sigma A \to \mathbb{C}$ of evaluation at 1. Here S^1 is identified with $\{\zeta \in \mathbb{C} : |\zeta| = 1\}$. (Note that ev_1 makes sense as a homomorphism to $\mathbb{C} since f(1) \in \mathbb{C} \cdot 1$ for $f \in \Sigma A$.)

Note that $\Sigma(A^+) = (SA)^+$, where SA is the conventional suspension $C_0(\mathbb{R} \otimes A$. A left adjoint for Σ in the pointed category immediately gives a left adjoint for S in the category of pro-C^*-algebras and arbitrary homomoprhisms, simply by taking the kernel of the homomorphism to $\mathbb{C}$ which comes with the pointed pro-C^*-algebra. It is a left adjoint for S that Rosenberg really had in mind, since what he calls Σ we call S. However, our approach more closely matches the topologists' conventions.

Definition 13.23. Let (A, α) be a pointed pro-C^*-algebra. We construct a pointed pro-C^*-algebra ΩA in terms of generators and relations as follows. Let the generating set G consist of the symbols $z(a, \zeta)$ for a $a \in A$ and $f \in S^1$, and let the relations R be as follows:

1. The map $(a, \zeta) \mapsto z(a, \zeta)$ continuous.
2. For each fixed $\zeta \in S^1$, the elements $z(a, \zeta)$ satisfy all the algebraic relations satisfied by the corresponding elements of A.
3. $z(1, \zeta) = z(1, 1) \ \forall S^1$.
4. $z(a, 1) = z(\alpha(a)1, 1) \ \forall \in A$.

Then set $\Omega A = C^*(G, R)$. The required homomorphism from ΩA to $\mathbb{C}$ is given by $z(a, \zeta) \mapsto \alpha(a)$ for $a \in A$ and $\zeta \in S^1$.

Theorem 13.24. *The map $\Phi : Hom_+(\Omega A, B) \to Hom_+(A, \Sigma B]$ defined by $\Phi(\varphi)(a)(\zeta) = \phi(z((a, \zeta))))$, is a natural bijection, and also defines a natural bijection $[\Omega A, B]^+ \to [A, \Sigma B]^+$. In particular, Ω is a left adjoint for the functor Σ.*

Proof. The statement about homotopy classes follows from the statement about homomorphisms on replacing B by $B \otimes C([0,1])$. Therefore we only consider the statement about homomorphisms. It is sufficient to prove that Φ defines a one to one correspondence between pointed homomorphisms from A to ΣB and representations ρ in B of (G, R) which correspond to pointed homomorphisms from ΩA to B. If $\beta : B \to C$ is the homomorphism making B a pointed algebra, then the conditions on ρ are $\rho(z(1,1)) = 1$ and $\beta \circ \rho(z(a,\zeta)) = \alpha(a)$ for $a \in A$ and $\zeta in S^1$. Using the universal property defining an inverse limit on the isomorphism $B = \lim_{p \in S(\mathrm{Ker}}(\beta))(\mathrm{Ker}(\beta)_p)^+$, and conditions (3) and (4) of the definition of weak
admissibility (Definition 13.19), we see that it suffices to consider the case in which B is a C^*-algebra.
Let $\rho : G \to B$ be a representation of (G, R) such that $\rho(z(1,1)) = 1$ and $\beta\rho(z(a,\zeta)) = \alpha(a)$. Define $\psi : A \to \Sigma B$ by $\psi(a)(\zeta) = \rho(z(a,\zeta)$. Then $\psi : A \to \Sigma B$ is certainly a unital *-homomorphism satisfying $ev_1(\psi(a)) = \rho(z(a,1)) = \alpha(a)$, as desired. (Note that $\psi(a) \in \Sigma B$, by the relations (3) and (4).) That ψ is continuous follows from relation (l) and the compactness of S^l : if $a_i \to a$ then the joint continuity of $\rho : A \times S^1 \to B$ forces $\rho(z(a_i,\zeta))$ to converge uniformly in ζ to $\rho(z(a,\zeta))$. This shows that the definition of Φ makes sense. Conversely, let $\psi : A \to \Sigma B$ be a pointed morphism. Define $p(z(a,\zeta)) = \psi(a)(\zeta)$. Clearly $\rho(z(1,1)) = 1$ and $\beta \circ \rho((z(a,\zeta)) = \beta(\psi(a)(\zeta) = \beta(\psi(a)(\zeta) = \alpha(a)$. Also, the relations of the definition all hold: (l) because ψ is continuous for the supremum norm on ΣB; (2) because evaluation at ζ composed with ψ is a homomorphism for all ζ; (3) because ψ is unital; and (4) because $\psi(a) = \beta(\psi(a)(1)) \cdot 1$ for $a \in A$.

□

13.25. It is an analog Ω_0 of a loop algebra construction without basepoint. Letting SB usual suspension of pro-C^*-algebra B, and identifying it with $\{f : S^1 \to B \mid f(1) = 0\}$ we have

$$\mathrm{Hom}(\Omega_0 A, B) \simeq \mathrm{Hom}(SA, B),$$

$$[\Omega_0 A, B] \simeq [A, SB].$$

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