Fundamental groups of C^* - algebras

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Abstract

Gelfand - Naimark theorem provides contravariant functor from category of commutative C^* - algebras to category of locally compact Hausdorff spaces. So category of (noncommutative) C^* algebras can be regarded as generalized (noncommutative) locally compact Hausdorff spaces. A set of topological invariants can be defined by algebraic methods. For example Serre Swan theorem [8] states that topological K - theory is in fact K - theory of C^* - algebras. However algebraic topology have rich set of invarnts. Some invariants have noncommutative generalizations. This article contains several steps towards definition of noncommutative generalization of fundamental group.

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1 Motivation. Preliminaries

In this preliminary section bridge (or mapping) between algebra and topology is considered. The Gelfand - Naimark theorem [10] can be thought of as the construction of two contravariant functors (cofunctors for short) from the category of locally compact Hausdorff spaces to the category of C^*-algebras. The first cofunctor C takes a compact space X to the C^*-algebra $C(X)$ of continuous complex-valued functions on X, and takes a continuous map $f : X \to Y$ to natural *-homomorphism $Cf : h \to hf : C(Y) \to C(X)$. If X is not compact space, but locally compact then corresponding C^* -algebra is $C_0(X)$ whose elements are continuous functions vanishing at infinity, and we require that the continuous maps $f : X \to Y$ be proper (the preimage of a compact set is compact) in order that $h \mapsto h \circ f$ take $C_0(Y)$ into $C_0(X)$. So (noncommutative) C^*-algebra can be considered as noncommutative generalization of locally compact Hausdorff space. Otherwise there exists inverse functor M that sets to any commutative C^*-algebra A a topological set of its characters $M(A)$. A lot of topological results related to locally compact spaces has its (noncommutative) algebraic analogues. Following picture presents bridge (mapping) between algebra and topology.

TOPOLOGY	ALGEBRA
Locally compact space	C^* - algebra
Compact space	Unital C^* - algebra
Continuous map	*-homomorpfism
Minimal compactification	Unitization
Maximal compactification	Algebra if multpicators
Closed subset	Ideal
Morphism of covering	?
Pointed space (X, x_0)	?
Fundamental group	?
Singular homology	?
Hurewicz homomorphism	?

If we consider category C^* - algebras then * - homomorphism is good analogue of continuous map. However * - homomorphisms are not good homomorphisms in general noncommutative case (See [13], [12]). Besides *- homomorphisms there are other types of morphisms of C^* - algebras. Morita equivalence and Kasparov intersection product [6] are examples of another types of morphisms. These new morphisms include *- homomorphisms. Isomorphisms can be considered as particular case of Morita equivalence and every *- homomorphisms defines unique Kasparov intersection product. Otherwise Morita equivalence does not provide substantial new results for commutative C^*-algebras. However a lot of results related to theory of C^*- algebras are impossible without Morita equivalence. A generalization of *- homomorphism is used for definition of fundamental group. This generalization uses some ideas related to Morita equivalence.

1.1 Different definitions of fundamental group

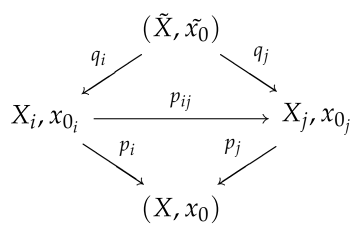
There exist a set of definitions of fundamental group. Algebraic topology [11] uses definition based on closed paths. However this definition do not have noncommutative generalization because noncommutative geometry is "The point is that there is no point". Noncommutative geometry does not have closed paths since it does not have no even points. Fundamental group can also be defined as covering group of universal covering. If topological space X is locally path connected and semilocally 1-path connected [11] then exist universal covering projection $\tilde{X} \to X$. In this case fundamental group is covering group of universal covering projection. Now it is not known what is noncommutative version of semilocally 1-path connected space. Even it is not known whether reasonable version of semilocally 1-path connected space exist. Even if this problem is eliminated we should define analogue of universal covering projection. We also do not know whether reasonable universal covering projection exist. But even if we do not have However there are a set of quite good noncommutative versions of finitely covering projection [3], [4] which are called Galois extensions. General universal covering projection is not finitely listed. However some constructions of infinite fundamental group are based on analogues of finitely listed covering projections. For example algebraic geometry also does not have good analogue of infinitely listed covering projection. However generalization of fundamental group can be constructed as inverse limit of finite coverings group. This construction is described in [9]. If case of algebraic manifold X over $\mathbb{C}$ this construction provides profinite completion of topological fundamental group i.e.

$$\pi_1^{alg}(X) = \widehat{\pi_1^{top}(X)},$$

where $\pi_1^{alg}(X)$ is fundamental group obtained by pure algebraic methods; $\pi_1^{top}(X)$ is fundamental group of X as Hausdorff space; symbol $\widehat{}$ means profinite completion. More details about this construction are considered in next subsection.

1.2 Fundamental group as inverse limit of finite groups

Algebraic geometry do not have good analogue of infinitely listed covering projections. However étale morphism [9] is good analogue of finitely listed covering projection. Complex algebraic manifold can be considered as algebraic manifold and as Hausdorff path connected and semilocally 1-path connected topological space. Étale morphisms of complex manifolds are topological finitely generated covering projections and vice versa. But infinitely listed covering projections cannot be algebraically defined. So étale morphisms cannot define infinite fundamental group. However étale morphisms provide determination of its profinite completion. The idea of this determination is explained below. Let (X, x_0) be locally path connected and semilocally 1-path connected pointed space [11]. Then there is universal pointed covering projection $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$. Then there exist group $G(\tilde{X}|X)$ of covering transformations[11]. By definition group of covering transformations is a group of such homeomorphisms of $\tilde{X}$ that every $g \in G(\tilde{X}|X)$ match following condition $p \circ g = p$. There is category of all normal covering projections of X and $\tilde{X}$ is universal object of this category. Following commutative diagram represents fragment of this category.



All maps in diagram (1.2) are pointed covering projections and $p_i \circ q_i = p_j \circ q_j = p$. It is covariant functor from category of covering projections of X to category of groups. Fragment of this category is presented below:

$$
\begin{array}{ccc}
 & G(\tilde{X}|X) & \\
{}^{q_i^*}\swarrow & & \searrow^{q_j^*} \\
G(X_i|X) & \xrightarrow{p_{ij}^*} & G(X_j|X) \\
{}_{p_i^*}\searrow & & \swarrow_{p_j^*} \\
 & \{e\} &
\end{array}
$$

Where $\{e\}$ is trivial group. All group homomorphisms of diagram (1.2) are surjective. Now suppose that we do not have $\tilde{X}$ but we have only such X_i that $G(X_i|X)$ is finite. Then we have following diagrams.

$$
\begin{array}{ccc}
(X_i, x_{0_i}) & \xrightarrow{p_{ij}} & (X_j, x_{0_j}) \\
{}_{p_i}\searrow & & \swarrow_{p_j} \\
 & (X, x_0) &
\end{array}
$$

$$
\begin{array}{ccc}
G(X_i|X) & \xrightarrow{p_{ij}^*} & G(X_j|X) \\
{}_{p_i^*}\searrow & & \swarrow_{p_j^*} \\
 & \{e\} &
\end{array}
$$

This diagram does not contain $G(\tilde{X}|X)$ however it contain important information about this group. Inverse limit of this diagram is called profinite completion of $G(\tilde{X}|X)$ [9]. So analogues of finite covering projections do not provide fundamental group. However they provide its profinite completion.

1.3 Galois extensions of associative algebras

Advanced definitions [5] of Galois extension use theory of Hopf algebras. However old definitions of Galois extensions are also useful for this paper.

Theorem 1.1. *[4] Let $\mathbb{F}$ be field,$\mathbb{E}/\mathbb{F}$ be a finite Galois extension of $\mathbb{F}$, $G = \mathrm{Gal}(\mathbb{E}/\mathbb{F})$. Then*

$$\mathrm{can} : \mathbb{E} \otimes_{\mathbb{F}} \mathbb{E} \to \mathrm{Map}(G, \mathbb{E}),$$

$$e_1 \otimes e_2 \mapsto (g \mapsto e_1 g(e_2)), \quad (e_1, e_2 \in \mathbb{E},\ g \in G)$$

is bijective.

Following definition can be regarded as noncommutative generalization of above theorem 1.1.

Definition 1.2. Let A, B be associative algebras over field k. An injective homomorphism $f : A \to B$ is a *Galois extension* if following conditions are hold:

1. $ga \in A$ $(\forall g \in \mathrm{Aut}(B), \forall a \in A)$.
2. It is split exact sequence
$$\{e\} \to G \to \mathrm{Aut}(B) \overset{\curvearrowleft}{\longrightarrow} \mathrm{Aut}(A) \to \{e\} \tag{1}$$
where G is kernel homomorphism $f : \mathrm{Aut}(B) \to \mathrm{Aut}(A)$, f is surjective and defined as
$$\mathrm{Aut}(B) \ni g \mapsto g|_A \in \mathrm{Aut}(A).$$
3. A is algebra of G invariants i.e $A = B^G$;
4. Canonical map
$$\mathrm{can} : B \otimes_A B \to \mathrm{Map}(G, B), \tag{2}$$
$$\sum_{i=1,\dots,n} b_i \otimes b'_i \mapsto (g \mapsto \sum_{i=1,\dots,n} b_i g(b'_i)), \ \ (b_i, b'_i \in B, \ \forall g \in G)$$
is bijective.

The G group is called *Galois group* of above extension. This extension is also called G - *Galois extension* for clarity. By $G(B|A)$ denote G.

Lemma 1.3. *Let $f : A \to B$ injective homomorphism of unital associative algebras, G acts on B. Homomorphism f satisfies condition 4 of definition 1.2 if there exists such elements $b_i, b'_i \in B$ ($i = 1, ..., n$) that following conditions are hold;*
$$\sum_{i=1,\dots,n} b_i b'_i = 1_B, \tag{3}$$
$$\sum_{i=1,\dots,n} b_i g b'_i = 0 \ \forall g \in G \ (g \text{ is nontrivial}). \tag{4}$$

Proof. 1. => By $e \in G$ denote unity of G. Let $f \in \mathrm{Map}(G, B)$ be such function that
$$f(e) = 1_B; \tag{5}$$
$$f(g) = 0; \ (g \neq e). \tag{6}$$
condition 4 of definition 1.2 means that:
$$B \otimes_A B \approx \mathrm{Map}(G, B).$$
Let $\sum_{i=1,\dots,n} b_i \otimes g b'_i \in B \otimes_A B$ be an element which correspond to f. It is clear that conditions (5), (6) are hold.

2. <= Let us enumerate elements of G, i.e $G = \{g_1, ..., g_{|G|}\}$. Let $f \in \mathrm{Map}(G, B)$ be a map from G to B; and $x \in B \otimes_A B$ is defined as
$$x = \sum_{i=1,\dots,|G|} f(g_i) b_i \otimes g_i^{-1} b'_i.$$
From (3), (4) it follows that $f = \mathrm{can}(x)$ (can is defined by equation (2).

□

Lemma 1.4. *Let A, B, C be associative algebras and $f : A \to B$, $h : B \to G$ Galois extension. Then composition $h \circ f : A \to B$ is Galois extension and it is following exact sequence of groups:*

$$\{e\} \to G(C|B) \overset{\curvearrowleft}{\longrightarrow} G(C|A) \overset{\curvearrowleft}{\longrightarrow} G(B|A) \to \{e\}.$$

Proof. Let us consider A as subalgebra of C. According to condition 2 of definition 1.2 it is following sequence of split surjective homomorphisms

$$\{e\} \to \mathrm{Aut}(C) \overset{\curvearrowleft}{\longrightarrow} \mathrm{Aut}(B) \overset{\curvearrowleft}{\longrightarrow} \mathrm{Aut}(A) \to \{e\}.$$

From above sequence it follows that

$$\{e\} \to G \overset{\curvearrowleft}{\longrightarrow} \mathrm{Aut}(C) \overset{\curvearrowleft}{\longrightarrow} \mathrm{Aut}(A) \to \{e\}$$

where G is kernel of $\mathrm{Aut}(C) \to \mathrm{Aut}(A)$. Above sequence is sequence of (1) So extension $g \circ f : A \to B$ satisfy condition 2 of definition 1.2. Also one can prove that:

$$\{e\} \to G(C|B) \overset{\curvearrowleft}{\longrightarrow} G \overset{\curvearrowleft}{\longrightarrow} G(B|A) \to \{e\}. \tag{7}$$

Condition 3 of definition 1.2 is clear. Let us check condition 4 of definition 1.2, i.e. $C \otimes_A C \approx \mathrm{Map}(G, C)$. According to 1.3 there exist such elements $b_i, b'_i \in B$ $(i = 1, ...m)$ and $c_j, c'_j \in B$ $(j = 1, ...n)$ that following conditions are hold:

$$\sum_{i=1,\dots,m} b_i b'_i = 1_B;$$

$$\sum_{i=1,\dots,m} b_i g' b'_i = 0 \ \forall g' \in G(B|A) \ (g' \text{ is nontrivial});$$

$$\sum_{j=1,\dots,n} c_j c'_j = 1_C;$$

$$\sum_{j=1,\dots,n} c_j g'' c'_j = 0 \ \forall g'' \in G(C|B) \ (g'' \text{ is nontrivial}).$$

By x_{ij} (resp x'_{ij}) denote $c_j b_i$ (resp $(b'_i c'_j)$. From direct proof it follows that

$$\sum_{i=1,\dots,m;\ j=1,\dots,n} x_{ij} x'_{ij} = 1_C;$$

Let $g \in G$ be nontrivial element, and $g|_B \in G(B|A)$ is also nontrivial.

$$\sum_{i=1,\dots,m;\ j=1,\dots,n} x_{ij} g x'_{ij} = \sum_{j=1,\dots,n} c_j \left(\textstyle\sum_{i=1,\dots,m} b_i g|_B b'_i\right) g c'_j = \sum_{j=1,\dots,n} c_j \cdot 0 \cdot g c'_j = 0.$$

Let us consider alternative case, $g \in G$ is nontrivial however $g|_B \in G(B|A)$ is trivial i.e. $g|_B \neq 1_B$. From exact sequence (7) it follows that $g \in G(C|B)$. So we have

$$\sum_{i=1,\dots,m;\ j=1,\dots,n} x_{ij} g x'_{ij} = \sum_{j=1,\dots,n} c_j \left(\textstyle\sum_{i=1,\dots,m} b_i g|_B b'_i\right) g c'_j = \sum_{j=1,\dots,n} c_j \cdot 1 \cdot g c'_j = \sum_{j=1,\dots,n} c_j g c'_j = 0.$$

We have proven that:

$$\sum_{i=1,...,m;\ j=1,...,n} x_{ij} g x'_{ij} = 0;\ \forall g \in G\ (g \text{ is nontrivial}).$$

According to lemma 1.3 condition 4 of definition 1.2 is hold. □

Using above lemma we can rephrase definition of G-Galois extension.

Definition 1.5. [3] Let A be unital directly indecomposable ring, G a finite group of automorphisms of a and $B = A^G$ a G - invariant subring. We call $f : B \to A$ G - *Galois extension* if there are elements $a_1, ..., a_n; b_1, ..., b_n$ in A such that $\sum_i a_i b_i = 1$; and $\sum_i (g a_i) b_i = 0$; for any nontrivial element $g \in G$.

Definition 1.6. If $f : B \to A$ is G - Galois extension and centralizer of B in A coincides with center of A then the extension is called outer G - Galois extension.

1.4 Galois extensions of C^* - algebras. Basic samples

In this section several Galois extensions of (non)commutative C^* - algebras are considered. First of all we shall consider commutative case.

Lemma 1.7. *If X is compact space and $p : Y \to X$ is finitely listed covering projection then $C(f) : C(Y) \to C(X)$ is Galois extension.*

Proof. By definition of covering every point $x \in X$ has such open connected neighborhood $U \subset X$ that $p^{-1}(U)$ is disjoint union of connected sets $\coprod V_j$ and $p|_{V_j} : V_j \to U$ is homeomorphism for every V_j. Since X compact we can select such finite covering U_i $(i = 1, ... < k)$ of X $(X = \bigcup U_i)$ that $p^{-1}(U_i) = \coprod V_j$ and $p|_{V_{ij}} : V_{ij} \to U_i$ is homeomorphism. Number of sets V_{ij} is finite because covering projection is finitely listed and $(j = 1, ..., l)$ where l is number of lists. There is partition of unity $\sum e_{ij} = 1_{C(Y)}$ subordinated to covering $Y = \bigcup V_{ij}$. If $g \in G(Y|X)$ is nontrivial element then $gV_{ij} \bigcap V_{ij} = \varnothing$ and $(g\sqrt{e_{ij}})\sqrt{e_{ij}} = 0$. Let us set $a_m = b_m = \sqrt{e_{ij}}$ $(m = il + j)$. Using previous formulas we have $\sum_m a_m b_m = 1_{C(Y)}$ and $\sum_m (g a_m) b_m = 0$. □

Definition 1.8. Let A, B be C^* - algebras; $f : A \to B$ is *-homomorphism. If f is G-Galois extension (see definition 1.2); G is finite group: then f is called *finite covering*; G is *covering group*.

Example 1.9. *Covering of noncommutative torus.* Let us consider Galois extensions of noncommutative torus. Noncommutative torus A_θ is C^* - norm completion of algebra generated by two unitary elements u, v and following conditions are hold:

$$uu^* = u^*u = vv^* = v^*v = 1;$$

$$uv = e^{2\pi i\theta} vu.$$

where $\theta \in \mathbb{R}$. If $\theta = 0$ then $A_\theta = A_0$ is commutative algebra of continuous functions on commutative torus $C(S^1 \times S^1)$ There is such trace τ_0 on A_θ that $\tau_0(\sum_{-\infty<i<\infty,-\infty<j<\infty} a_{ij} u^i v^j) =$

a_{00}. C^* - norm of A_θ is defined by following way $\|a\| = \sqrt{\tau_0(a^*a)}$. Let us consider * - homomorphism $f : A_\theta \to A_{\theta'}$ where $A_{\theta'}$ is generated by unitary elements u' and v'. Homomorphism f is defined by following way:

$$u \mapsto u'^m;$$

$$v \mapsto v'^n;$$

It is easy to check that $\theta' = \frac{\theta+k}{mn}$ $(k = 0, ..., mn-1)$. Let us show that *-homomorphism f is Galois extension. First of all note that commutative $C*$- subalgebras $C(u') \subset A_{\theta'}$ and $C(v') \subset A_{\theta'}$ generated by u' and v' respectively are isomorphic to algebra $C(S^1)$ where S^1 is one dimensional circle. There are induced by f *-homomorphisms $C(S^1) = C(u) \to C(u') = C(S^1)$, $C(S^1) = C(v) \to C(v') = C(S^1)$. These *-homomorphisms induces m and n listed covering projections respectively. Covering groups of these covering projections are $G_1 \approx \mathbb{Z}_m$ and $G_2 \approx \mathbb{Z}_n$ respectively. Generators of these groups are presented below:

$$u' \mapsto e^{\frac{2\pi i}{m}} u';$$

$$v' \mapsto e^{\frac{2\pi i}{n}} v'.$$

Since homomorphisms of commutative algebras $C(u) \to C(u')$, $C(u) \to C(u')$ correspond to covering projections then according to 1.7 there are such elements x_i $(i = 1, ..., r)$, y_j $(j = 1, ..., s)$ that.

$$\sum_{1 \le i \le r} x_i^2 = 1_{C(u')};$$

$$\sum_{1 \le i \le r} (g_1 x_i) x_i = 0; g_1 \in G_1;$$

$$\sum_{1 \le j \le s} y_i^2 = 1_{C(v')};$$

$$\sum_{1 \le j \le s} (g_2 y_i) y_i = 0; g_2 \in G_2.$$

where g_1 and g_2 are nontrivial elements of G_1 and G_2.
Actions of G_1 and G_2 induce action of $G = G_1 \times G_2$ on $A_{\theta'}$. Let us set

$$a_k = y_j x_i;$$

$$b_k = x_i y_j;$$

where $k = im + j$.
It is easy to check following equalities.

$$\sum_{1 \le k \le mn} a_k b_k = 1_{A_{\theta'}};$$

$$\sum_{1 \le k \le mn} (g a_k) b_k = 0.$$

where $g \in G$ is nontrivial element.
Above equations are conditions 1.5 of G - Galois extension. Let us try to construct analogue of fundamental group as inverse limit groups of Galois extensions. If $\theta = 0$ then $A_\theta = A_0$ is C^* - algebra of commutative torus ($A_0 = C(\mathbb{T}^2)$). For all $m, n, k \in \mathbb{N}$ there is $Z_m \times Z_n$ extension $f : A_0 \to A_{\frac{k}{mn}}$ of noncommutative torus which is defined by following rules.

$$u \mapsto u'^m; v \mapsto v'^n. \tag{8}$$

If $k = 0$ then extension is commutative. Otherwise extension is noncommutative. So there are noncommutative Galois extensions of A_0 besides commutative ones.

1.5 Homomorphism is not good generalization of covering

We have obtained additional noncommutative G - Galois extensions of commutative C^* - algebra. So if we consider direct generalization of above construction of fundamental group as inverse limit then we obtain too big group. Reduction of fundamental group can be obtained if we make equivalence of some coverings.

Example 1.10. *Coverings of commutative torus.* There is a set of reasons to make Galois extension $f_1 : A_0 \to A_{\frac{k_1}{mn}}$ equivalent to $f_2 : A_0 \to A_{\frac{k_2}{mn}}$ ($k_1 \neq k_2$). These extensions are defined by following equations.

$$u \mapsto u_1^m; v \mapsto v_1^n; \tag{9}$$

$$u \mapsto u_2^m; v \mapsto v_2^n. \tag{10}$$

where $u, v \in A_0$, $u_1, v_1 \in A_{\frac{k_1}{mn}}$, $u_2, v_2 \in A_{\frac{k_2}{mn}}$ are unitary generators of C^* - algebras. Homomorphisms f_1 and f_2 are defined by equations (9) and (10) respectively. Let us consider reasons for making equivalent these extensions. Covering projection of commutative torus $p : S^1 \times S^1 \to S^1 \times S^1$ is completely defined by winding numbers m, n of first and second component of direct product $S^1 \times S^1$. These winding numbers are degrees of coverings of S^1. It is evident analogy between these numbers and numbers if Galois extensions defined by (9) and (10). Second reason is that every Galois extension $f_2 : A_0 \to A_{\frac{k}{mn}}$ becomes equivalent to commutative covering $f : A_0 \to A_0$. If every noncommutative covering of commutative torus is equivalent to commutative one then fundamental group fundamental group coincides to (profinite completion) of classical fundamental group. Equivalence can be obtained by extension of notion of covering morphism. There are a set of extensions (generalizations) of *- homomorphism. For example instead * - homomorphisms $A \to B$ homomorphisms $A \to B \otimes \mathcal{K}$ ($\mathcal{K}$ is algebra of compact operators) could be considered [2]. Also one can consider homomorphisms to matrix algebras $A \to \mathbb{M}_n(B) \approx B \otimes \mathbb{M}_n(\mathbb{C})$. Let us show how morphisms to matrix algebras could be used for equivalence of Galois extensions $f_1 : A_0 \to A_{\frac{k_1}{mn}}$ and $f_2 : A_0 \to A_{\frac{k_2}{mn}}$ ($k_1 \neq k_2$) defined by equations (9) and (10). Let us select matrixes $w_1, w_2 \in \mathbb{M}_{mn}(\mathbb{C})$ that

$$w_1^{mn} = w_2^{mn} = 1;$$

$$w_1 w_2 = e^{2\pi i \frac{k_2 - k_1}{mn}} w_2 w_1.$$

Using these matrixes we can define homomorphism $A_{\frac{k_1}{mn}} \to A_{\frac{k_2}{mn}} \otimes \mathbb{M}_{mn}$ defined by following way:

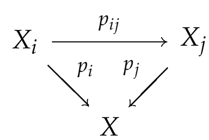
$$u_1 \mapsto u_2 \otimes w_1;$$

$$v_1 \mapsto v_2 \otimes w_2;$$

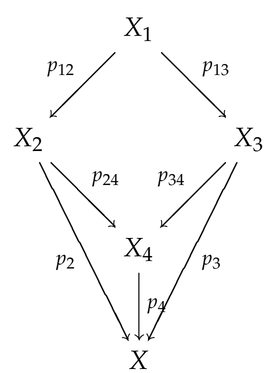
This homomorphism can make equivalence of f_1 and f_2. Homomorphism from $A_{\frac{k_1}{mn}}$ to $A_{\frac{k_2}{mn}}$ does not always exist. However from homomorphism $A_{\frac{k_1}{mn}}$ to $A_{\frac{k_2}{mn}} \otimes \mathbb{M}_{mn}$ exists always. These additional homomorphisms can make necessary equivalences.

1.6 Why one need base point

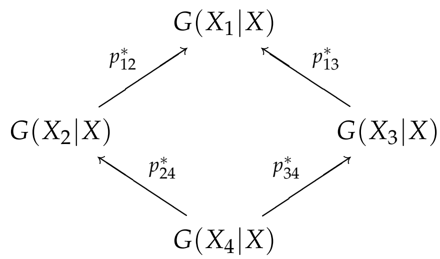
In previous text fundamental group by coverings of pointed spaces. Noncommutative geometry does not have points and so it does not have base points. So we need analogue of base point. Before looking for these analogues we need understand purpose of base point. What will occur if we drop base points. In these case diagram 1.2 will be replaced by following diagram.



However this diagram should not be commutative. For example this diagram can contain following fragment.



It is possible that $p_{24} \circ p_{12} \neq p_{24} \circ p_{12}$ because coverings $p_{24} \circ p_{12}$ and $p_{24} \circ p_{12}$ are coverings which are roughly speaking coverings with different base points. In this case following diagram of groups should not be commutative.



So one cannot use construct fundamental group as inverse limit. However if all groups of above diagram are Abelian then above diagram is commutative.

1.7 Groupoids

Definition 1.11. [14]. A *groupoid* consists of a set G, a distinguished subset $G^{(0)}$, two maps $r, s : G \to G^{(0)}$ and a law of composition

$$\circ : G^{(2)} = \{(\gamma_1, \gamma_2) \in G \times G; s(\gamma_1) = r(\gamma_2)\} \to G.$$

such that

1. $s(\gamma_1 \circ \gamma_2) = s(\gamma_2);\ r(\gamma_1 \circ \gamma_2) = r(\gamma_1)\ \forall (\gamma_1, \gamma_2) \in G^{(2)}$
2. $s(x) = r(x) = x\ \forall x \in G^{(0)}$
3. $\gamma \circ s(\gamma) = \gamma;\ r(\gamma) \circ \gamma = \gamma\ \forall \gamma \in G$
4. $(\gamma_1 \circ \gamma_2) \circ \gamma_1 = \gamma_3 \circ (\gamma_2 \circ \gamma_3)$
5. Each γ has a two-sided inverse γ^{-1}, with $\gamma \circ \gamma^{-1} = r(\gamma)\ ;\ \gamma^{-1} \circ \gamma = s(\gamma)$.

Remark 1.12. Groupoid can be considered as (small) category where G is its set of arrows $G^{(0)}$ is set of identical arrows (or objects) and $\circ$ is composition law. All morphisms of groupoid category are isomorphisms.

Definition 1.13. [14]. A *smooth groupoid* G is a groupoid together with a differentiable structure on G and $G^{(0)}$ such that the maps r and s are submersions, and the object inclusion map $G^{(0)} \to G$ is smooth, as is the composition map $G^{(2)} \to G$.

1.8 Morita Equivalence

Theory of C^* - algebras has a set of different type of morphisms and *- homomorphism is particular case only. Morita equivalence can be considered as homomorphism of (C^*) - algebras. Modification of Morita equivalence is considered in this article below.

Definition 1.14. [15] Algebra A is *Morita equivalent* to algebra B if it is such $A - B$ bimodule E that:

1. $E \otimes_B E^* \approx A$ as $A - A$ bimodule;

2. $E^* \otimes_A E \approx B$ as $B - B$ bimodule

where

$$E^* = \mathrm{Hom}_A(E, A).$$

Example 1.15. *Morita equivalence of noncommutative torus.* Here we show that C^* - algebra A_θ of noncommutative torus is Morita equivalent to $A_{\theta+q/p}$ where $p, q \in \mathbb{N}$ and p, q are relatively prime numbers. First of all we introduce pre - C^* - algebra $\mathcal{A}_\theta$ [17]. As well as A_θ algebra $\mathcal{A}_\theta$ is generated by two unitary elements u and v. Indeed $\mathcal{A}_\theta$ is subalgebra of A_θ. Every $a \in \mathcal{A}_\theta$ can be represented by following way:

$$a = \sum_{ij} a_{ij} u^i v^j$$

and there is such $C \in \mathbb{R}$ that
$\exists C \in \mathbb{R}_+ (1 + i^2 + j^2)^k a_{i,j} < C$ for any $i, j \in \mathbb{Z}$, $k \in \mathbb{N}$.
Algebra $\mathcal{A}_\theta$ can be interpreted as convolution algebra of smooth foliation groupoid [14]. Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be two dimensional torus and F is irrational foliation. Every leave of foliation is defined by following equations:

$$y = y_0 + \theta t \equiv \text{modulo } 1;\ (t \in R)$$

$$x = x_0 + t \equiv mod\text{modulo } 1$$

where $x, y \in \mathbb{R}/\mathbb{Z}$ are coordinates of torus. The foliation groupoid G is groupoid of homotopy equivalence classes of paths which belong to leaves of foliation. Let $N_{0,1} \subset \mathbb{T}^2$ be submanifold defined by equation $x = 0$. The smooth groupoid $G_{N_{0,1}}$ is groupoid of classes of homotopy equivalent classes of paths. Every path belong to single leaf of foliation. Either begin and end of path belongs to N. The space of leaves is set which parameterized by pairs $(y_0, n) \in S^1 \times \mathbb{Z}$ where y_0 is y coordinate of begin of path and $y_0 + n\theta \equiv$ modulo 1 is y - coordinate of end of path. So $G_{N_{0,1}}$ can be considered as disjoint union if countable set of circles. Differential structure of G_N is induced by differentiable structure of circle. Convolution algebra is $C_0^\infty(G_N)$ as vector space. Composition law is defined by following expression.

$$ab(\gamma) = \sum_{\gamma_1 \circ \gamma_2 = \gamma} a(\gamma_1) b(\gamma_2);\ \forall \gamma_1, \gamma_2 \in G_N. \tag{11}$$

Let us consider unitary generators $u, v \in C_0^\infty(G_N)$. First generator is defined by following way:

1. If $s(\gamma) \neq r(\gamma)$ then $u(\gamma) = 0$;

2. If $s(\gamma) = r(\gamma) = (0, y) \in \mathbb{T}^2$ then $u(\gamma) = e^{2\pi i y}$;

Second operator satisfies following condition

1. If $s(\gamma) = (0, y)$ & $r(\gamma) = (0, y + \theta)$ then $v(\gamma) = 1$;
2. Otherwise $v(\gamma) = 0$;

Composition uv complies following condition.

1. If $s(\gamma) = (0, y)$ & $r(\gamma) = (0, y + \theta)$ then $uv(\gamma) = e^{2\pi i(y+\theta)}$;
2. Otherwise $uv(\gamma) = 0$;

and composition vu complies following conditions.

1. If $s(\gamma) = (0, y)$ & $r(\gamma) = (0, y + \theta)$ then $vu(\gamma) = e^{2\pi iy}$;
2. Otherwise $uv(\gamma) = 0$;

So we have:

$$vu = e^{2\pi i\theta} uv.$$

This relation between generators of $C_0^\infty(G_{N_{0,1}})$ coincides with relation between generators of $\mathcal{A}_\theta$. Indeed $\mathcal{A}_\theta \approx C_0^\infty(G_{N_{0,1}})$ Now let us consider new submanifold $Np, q = \{(ps\text{modulo}1, qs\text{modulo}1) \in \mathbb{T}^2 | p, q$ are relatively prime integers $s \in \mathbb{R}$. Using similar to above calculations we can establish following isomorphism:

$$\mathcal{A}_{\theta - p/q} \approx C_0^\infty(G_{N_{p,q}})$$

Now we would like to find Morita equivalence between $C_0^\infty(G_{N_{0,1}})$ and $C_0^\infty(G_{N_{p,q}})$ First of all let us define smooth manifold

$$E_{p,q} = \{\gamma \in G;\ r(\gamma) \in N_{p,q},\ s(\gamma) \in N_{0,1}\}.$$

Let us assume that $p > 0$ to avoid the trivial case $p = 0$. One finds then that $E_{p,q} = \{(0, y), t);\ t \in \mathbb{R},\ py = t(q - p\theta)\ mod\text{modulo } 1\}$. It is thus the disjoint union of p copies of the manifold $\mathbb{R}$, since the value of $y \in \mathbb{R}/\mathbb{Z}$ is not uniquely determined by the equality

$$py = t(q - p\theta)\ mod\text{modulo } 1.$$

It is left (resp right) action of $N_{0,1}$ (resp. $N_{p,q}$) on $E_{p,q}$. These actions are defined as composition of paths in G. Let us define following module $\mathcal{E}_{p,q} = C_0^\infty(E_{p,q})$. Left (resp. right) action of groupoid $G_{N_{0,1}}$ (resp. $G_{N_{p,q}}$) on $E_{p,q}$ induces left (resp right) action of $C_0^\infty(G_{N_{0,1}})$ (resp $C_0^\infty(G_{N_{p,q}})$) on $\mathcal{E}_{p,q}$. Explicit formula of right action of $C_0^\infty(G_{N_{p,q}})$ is presented below:

$$(xa)(\eta) = \sum_{\eta \circ \zeta = \gamma} x(\eta) a(\zeta);\ \forall x \in \mathcal{E}_{p,q};\ \forall a \in C_0^\infty(G_{N_{p,q}}).$$

Left action of $C_0^\infty(G_{N_{p,q}})$ can be defined by similar way. It is useful to define alternative representations of actions. First of all we define isometric covering $f' : \mathbb{R} \to G_{N_{p,q}}$ which satisfy following condition:

$$f'(n) = (0, 0) \in \mathbb{T}^2;\ \forall n \in \mathbb{Z}.$$

Let $\gamma_0 \in E_{p,q}, r(\gamma_0) = s(\gamma_0) = (0,0) \in \mathbb{R}^2/\mathbb{Z}^2$. Let $E^0_{p,q} \subset E_{p,q}$ be such path - connected subset that $\gamma_0 \in E_{p,q_0}$. This subset is diffeomorphic to $\mathbb{R}$ as differentiable manifold. Let $f : \mathbb{R} \to \mathbb{R}^2/\mathbb{Z}^2$ be defined as $t \mapsto (0, t\text{modulo}1)$. We have differentiable parametrization $f' : \mathbb{R} \to E^0_{p,q}$ which satisfies following conditions:

1. $f''(0) = \gamma_0$;
2. If $\gamma = f''(t)$ then $r(\gamma) = f'(t)\ \forall t \in \mathbb{R}$.

We can obtain other path connected components $E^1_{p,q}$, ..., $E^{p-1}_{p,q}$ using join of paths. For example if $\gamma \in E^0_{p,q}$ and $\eta \in G_{N_{0,1}}$ is such path that $r(\eta) = s(\gamma)$ and η contains is one circle path then $\eta \circ \gamma \in E^1_{p,q}$. Similarly join of $2, ..., p-1$ join paths provide $E^2_{p,q}$, ..., $E^{p-1}_{p,q}$ respectively. Join p circle path provides $E^0_{p,q}$ components once again. So we have:

$$E_{p,q} = \bigsqcup_{i=0,...,p-1} E^i_{p,q};$$

$$\mathcal{E}_{p,q} = C^\infty_0(E_{p,q}) = \bigoplus_{i=0,...,p-1} \mathcal{E}^i_{p,q} \approx C^\infty_0(\mathbb{R}) \oplus \mathbb{C}^p.$$

Let $u', v' \in \mathcal{A}_\theta$ be canonical generators. Action of u', v' on $\mathcal{E}_{p,q} =\approx C^\infty_0(\mathbb{R}) \oplus \mathbb{C}^p$ is induced by following action on $C^\infty_0(\mathbb{R})$:

$$(u'f)(t) = e^{2\pi i t} f(t);\ (v'f)(t) = f(t + \theta - q/p);\ \forall f \in C^\infty_0(\mathbb{R}); \forall t \in \mathbb{R}. \tag{12}$$

Let $U, V \in \mathbb{M}_p$ be matrixes defined by following way:

$$U = \begin{pmatrix} 1 & ... & 0 \\ 0 & e^{2\pi i q/p} & 0 \\ ... & & \\ 0 & ... & e^{2\pi i q(p-1)/p} \end{pmatrix};\ V = \begin{pmatrix} 0 & 0 & ... & 0 & 1 \\ 1 & 0 & ... & 0 & 0 \\ 0 & 1 & ... & 0 & 0 \\ ... & & & & \\ 0 & 0 & ... & 1 & 0 \end{pmatrix}.$$

Matrix V performs cyclic transposition of vector $x \in \mathbb{C}^p$. One can easy check that:

$$U^p = V^p = 1;\ UV = e^{2\pi i q/p} VU.$$

Generators $u, v \in \mathcal{A}_\theta$ act on $C^\infty_0(\mathbb{R}) \oplus \mathbb{C}^p$ as $u' \otimes U$ and $v' \otimes V$ respectively. Since $C^\infty_0(\mathbb{R}) \oplus \mathbb{C}^p \approx \mathcal{E}_{p,q}$ we have induced action of $\mathcal{A}_{theta}$ on $\mathcal{E}_{p,q}$. Let us also introduce dual groupoid

$$E^*_{p,q} = \{\gamma \in G;\ r(\gamma) \in N_{0,1},\ s(\gamma) \in N_{p,q}\}.$$

Similarly $\mathcal{E}_{p,q} = C^\infty_0(E^*_{p,q})$ is $C^\infty_0(G_{N_{p,q}}) - C^\infty_0(G_{N_{p,q}})$ bimodule. We can establish following bimodule isomorphism

$$\mathcal{E}^*_{p,q} \approx \mathrm{Hom}_{C^\infty_0(G_{N_{0,1}})}(\mathcal{E}_{p,q}, C^\infty_0(G_{N_{0,1}})).$$

This homomorphism is defined by following way:

$$x \mapsto (y \mapsto a)$$

where

$$a(\gamma) = \sum_{\eta \circ \zeta = \gamma} x(\eta) y(\zeta).$$

Morita equivalence can be established as following bimodule isomorphisms.

$$\mathcal{E}_{p,q} \otimes \mathcal{E}^*_{p,q} \approx C^\infty_0(G_{N_{0,1}}),$$

$$\mathcal{E}^*_{p,q} \otimes \mathcal{E}_{p,q} \approx C^\infty_0(G_{N_{p,q}}).$$

These homomorphism are defined by following way:

$$x \otimes y \mapsto a;\ y \otimes x \mapsto b;\ x \in \mathcal{E}_{p,q},\ x \in \mathcal{E}_{p,q},\ a \in C^\infty_0(G_{N_{0,1}})\ b \in C^\infty_0(G_{N_{p,q}})$$

where a and b satisfy following conditions:

$$a(\gamma) = \sum_{\eta \circ \zeta = \gamma} x(\eta) y(\zeta);\ \forall \gamma \in G_{N_{0,1}},$$

$$b(\gamma') = \sum_{\zeta \circ \eta = \gamma'} y(\zeta) x(\eta);\ \forall \gamma' \in G_{N_{p,q}}.$$

So we have Morita equivalence between $C^\infty_0(G_{N_{0,1}})$ and $C^\infty_0(G_{N_{p,q}})$. These algebras are isomorphic to $\mathcal{A}_\theta$ and $\mathcal{A}_{\theta-q/p}$ respectively. Otherwise $C^\infty_0(G_{N_{0,1}})$ and $C^\infty_0(G_{N_{p,q}})$ are dense local subalgebras of C^* - algebras $C_r(G_{N_{0,1}})$, $C_r(G_{N_{p,q}})$ respectively. Theory of local subalgebras and their Morita equivalence is explained in [14], [6]. According to this theory algebras $C_r(G_{N_{0,1}})$, $C_r(G_{N_{p,q}})$ are also Morita equivalent. Moreover it is such norm on $\mathcal{E}_{p,q}$ that Morita equivalence between $C_r(G_{N_{0,1}})$, $C_r(G_{N_{p,q}})$ is implemented by norm completion $\overline{\mathcal{E}}_{p,q}$ of $\mathcal{E}_{p,q}$. Otherwise as it is shown in [14] A_θ (resp. $A_{\theta+q/p}$) is isomorphic to $C_r(G_{N_{0,1}})$ (resp. $C_r(G_{N_{0,1}})$) for any irrational θ. So we have Morita equivalence between A_θ and $A_{\theta+q/p}$.

2 Covering morphisms

Let us generalize notion of morphisms of covering projections. According to speculations of previous section following definition looks reasonable.

Definition 2.1. Let $p_1 : A \to A_1$ and $p_2 : A \to A_2$ be Galois extensions; G_1, (resp. G_2) is Galois group of p_1 (resp p_2); E is $A_1 - A_2$ bimodule; $g : G_2 \to G_2$ surjective group homomorphism. A pair (E, f) is called a *covering morphism* from p_2 to p_1 if the following conditions hold:

1. Let E^* be dual $A_2 - A_1$ bimodule defined by following way $E^* = \mathrm{Hom}_{A_1}(E, A_1)$. Then $A_2 \approx E^* \otimes_{A_1} E$ as A_2 bimodules;

2. If $x \in E$, $x^* \in E^*$, $a \in A$ then $p_1(a)x = xp_2(a)$, $p_1(a)x^* = x^*p_2(a)$.

3. Group G_2 acts on both E, E^*. If $x \in E$, $x^* \in E^*$, $g \in G$ then:

$$g(xa_2) = (gx)(ga_2); \tag{13}$$

$$g(a_2x^*) = (ga_2)(gx^*); \tag{14}$$

$$g(a_1x) = (f(g)a_1)(gx); \tag{15}$$

$$g(x^*a_1) = (gx^*)(f(g)a_1); \tag{16}$$

4. Group G_2 naturally acts on $E \otimes_{A_2} E^*$ since G_2 acts on either E and E^*. So group Let $G = ker\ f$ acts on $E \otimes_{A_2} E'$. Let $(E \otimes_{A_2} E^*)^G$ be submodule of G - invariants. It is following $A_2 - A_2$ bimodule isomorphism:

$$A_2 \approx (E \otimes_{A_2} E^*)^G;$$

Remark 2.2. This definition provides reduction of fundamental group (See section 1.5).

Lemma 2.3. *Let $p_1 : A \to A_1$, $p_2 : A \to A_2$, $p_3 : A \to A_3$ be covering morphisms; (E_1, f_{21}) (resp. (E_2, f_{32})) is covering morphisms from p_2 to p_1 (resp. from p_3 to p_2). Then*

$$(E_1 \otimes_{A_2} E_2, E_2^* \otimes_{A_2} E_1^*, f_{32} \circ f_{21})$$

is covering morphism from p_3 to p_1.

Proof. By G_1, G_2, G_3 denote groups of coverings p_1, p_2, p_3 respectively. Homomorphism $f_{32} \circ f_{21}$ is surjective because either f_{21} and f_{32} are surjective. Natural actions of G_3 on $E_1 \otimes_{A_2} E_2$ and $E_2^* \otimes_{A_2} E_1^*$ are defined by following way:

$$g \circ (x_1 \otimes x_2) = f_{32}(g)x_1 \otimes gx_2, (\ g \in G_3,\ x_1 \in E_1,\ x_2 \in E_2).$$

$$g \circ (x_2^* \otimes x_1^*) = gx_2^* \otimes f_{32}(g)x_1^*, (\ g \in G_3,\ x_1^* \in E_1^*,\ x_2' \in E_2^*).$$

We have

$$(E_2^* \otimes_{A_2} E_1^*) \otimes_{A_1} (E_1 \otimes_{A_2} E_2) =$$
$$= E_2^* \otimes_{A_2} (E_1^* \otimes_{A_1} E_1) \otimes_{A_2} E_2 \approx$$
$$\approx E_2^* \otimes_{A_2} A_2 \otimes_{A_2} E_2 = E_2^* \otimes_{A_2} E_2 \approx A_3.$$

Similarly we have following sequence of bimodule isomorphisms:

$$((E_1 \otimes_{A_2} E_2) \otimes_{A_3} (E_2^* \otimes_{A_2} E_1^*))^{ker(f_{32} \circ f_{21})} =$$
$$= (E_1 \otimes_{A_2} (E_2 \otimes_{A_3} E_2^*)^{ker(f_{32})} \otimes_{A_2} E_1^*)^{ker(f_{21})}$$
$$\approx (E_2 \otimes_{A_2} A_2 \otimes_{A_2} E_2)^{ker(f_{21})} = (E_2^* \otimes_{A_2} E_2^*)^{ker(f_{21})} \approx A_1.$$

□

Example 2.4. *Covering morphism of noncommutative torus*. Let us consider two coverings of noncommutative torus A_θ. Let u, v be are generators of A_θ. According to 1.9 if $m, n \in \mathbb{N}$ then it is such covering $p' : A_\theta \to A_{\frac{\theta+k}{mn}}$ which is defined by following way

$$u \mapsto u'^m;$$

$$v \mapsto v'^n$$

where $m, n \in \mathbb{N}$, $k \in \mathbb{Z}$ and u', v' be are generators of $A_{\frac{\theta+k}{mn}}$.
Galois group of this covering is $G' = \mathbb{Z}_m \times \mathbb{Z}_n$ which has two generators g'_1, g'_2 of periods m and n respectively. These generators act on $A_{\frac{\theta+k}{mn}}$ by following way:

$$g'_1 u' = u' e^{2\pi i/m};$$

$$g'_1 v' = v';$$

$$g'_2 u' = u';$$

$$g'_2 v' = v' e^{2\pi i/n}.$$

Let us consider another covering $p'' : A_\theta \to A_{\frac{\theta+l}{rmsn}}$ which is defined by following way:

$$u \mapsto u''^r m;$$

$$v \mapsto v''^r n$$

where $r, s \in \mathbb{N}$, $l \in \mathbb{Z}$ and u'', v'' be are generators of $A_{\frac{\theta+l}{rmsn}}$.
Galois group of this covering is $G'' = \mathbb{Z}_{rm} \times \mathbb{Z}_{sn}$ which has two generators g''_1, g''_2 of periods m and n respectively. These generators act on $A_{\frac{\theta+l}{rmsn}}$ by following way:

$$g''_1 u'' = u'' e^{2\pi i/rm};$$

$$g''_1 v'' = v'';$$

$$g''_2 u'' = u'';$$

$$g''_2 v'' = v'' e^{2\pi i/sn}.$$

Now let us construct covering morphism (E, f) from p'' to p'. Group homomorphism $f : G'' \to G'$ is defined by following way:

$$g''_1 \mapsto g'_1;$$

$$g''_2 \mapsto g'_2.$$

Construction of $A_{\frac{\theta+k}{mn}} - A_{\frac{\theta+l}{rmsn}}$ bimodule is more complicated. Let $\theta' = \frac{\theta+k}{rmsn}$ be and $\frac{\theta+l}{rmsn} = \theta' - q/p$ where q/p is irreducible fraction. Then according to 1.15 $\mathcal{A}_{\theta'}$ is Morita equivalent to $\mathcal{A}_{\theta'-p/q}$. This Morita equivalence is obtained by $\mathcal{A}_{\theta'} - \mathcal{A}_{\theta'-p/q}$ bimodule $\mathcal{E}_{p,q}$. Since $\frac{\theta+l}{rmsn} = \theta' - q/p$ right action of $\mathcal{A}_{\frac{\theta+l}{rmsn}}$ on $\mathcal{E}_{p,q}$ is evident. Left action of $\mathcal{A}_{\frac{\theta+k}{mn}}$ is defined as "composition" of homomorphism $f'' : \mathcal{A}_{\frac{\theta+k}{mn}} \to \mathcal{A}_{\theta'}$ and left action of $\mathcal{A}_{\theta'}$ on $\mathcal{E}_{p,q}$. Homomorphism f'' is defined by following way:

$$u \mapsto u''^{r};$$

$$v \mapsto v''^{s}.$$

where $u'', v'' \in \mathcal{A}_{\theta'}$ are canonical generators of $\mathcal{A}_{\theta'}$. Now let us define action of $G'' = \mathbb{Z}_{rm} \times \mathbb{Z}_{sn}$ on $\mathcal{E}_{p,q}$. Every function $g \in C_0^\infty(\mathbb{R})$ can be represented as:

$$f(t) = \sum_{a,b \in \mathbb{Z}} k_{a,b} e^{2\pi i a t} e^{-(t-b)^2}; \tag{17}$$

$$\forall a, b \in \mathbb{Z}, \forall N \in \mathbb{N} \ \exists C \in \mathbb{R}_+ \ (1 + a^2 + b^2)^N k_{a,b} < C.$$

Usage of above representation enable us define action of $G'' = \mathbb{Z}_{rm} \times \mathbb{Z}_{sn}$ on $\mathcal{E}_{p,q}$. Let $g_1 = (1 \ mod \text{modulo} \ rm, 0 \ mod \text{modulo} \ sn) \in \mathbb{Z}_{rm} \times \mathbb{Z}_{sn}$ and $g_2 = (0 \ mod \text{modulo} \ rm, 1 \ mod \text{modulo} \ sn) \in \mathbb{Z}_{rm} \times \mathbb{Z}_{sn}$ be generators of $G'' = \mathbb{Z}_{rm} \times \mathbb{Z}_{sn}$. Action of on $C_0^\infty(\mathbb{R})$ is defined as action on coefficient $k_{a,b}$ in (17) by following way:

$$g_1 k_{a,b} = k_{a,b} e^{2\pi i a / rm}; \tag{18}$$

$$g_2 k_{a,b} = k_{a,b} e^{2\pi i b / sn}.$$

Since $\mathcal{E}_{p,q} \approx C_0^\infty(\mathbb{R}) \otimes \mathbb{C}^p$ above action of G'' on $C_0^\infty(\mathbb{R})$ naturally induces action of G'' on $\mathcal{E}_{p,q}$. Direct calculation proves that this action comply to equations (13)-(16).

3 Generalization of fundamental group functor

Fundamental group functor[11] is a functor from category of topological sets to category of groups. This functor is defined by following way

$$X \ \mapsto \ \pi_1(X),$$

$$f : X \to Y \ \mapsto \ \pi_1(f) : \pi_1(X) \to \pi(Y).$$

Noncommutative generalization of fundamental group $\pi_1(X)$ is not known yet. However we know generalization of covering group. So one can construct generalization of $\pi_1(f)$ with respect to covering. First of all we define pi_1 with respect to covering in commutative case.

Let $f : X \to Y$ be continuous map, and $\widetilde{X} \to X$, $\widetilde{Y} \to Y$ such normal coverings that following diagram

$$\begin{array}{ccc} \widetilde{X} & \longrightarrow & \widetilde{Y} \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

This diagram induces following diagram with surjective vertical arrows.

$$\begin{array}{ccc} \pi_1(X) & \longrightarrow & \pi_1(Y) \\ \downarrow & & \downarrow \\ G(\widetilde{X}|X) & \longrightarrow & G(\widetilde{Y}|Y) \end{array}$$

Definition 3.1. Let us consider above coverings. Homomorphism $G(\widetilde{X}|X) \to G(\widetilde{Y}|Y)$ is called *Fundamental group homomorphism* with respect to coverings $\widetilde{X} \to X$, $\widetilde{Y} \to Y$.

This definition have noncommutative generalization.

Definition 3.2. Let A, B be C^* algebras, $f : A \to B$ * - homomorphism, $A \to \widetilde{A}$, $B \to \widetilde{B}$ coverings. Suppose that it is following commutative diagram:

$$\begin{array}{ccc} \widetilde{A} & \xrightarrow{\widetilde{f}} & \widetilde{B} \\ \uparrow & & \uparrow \\ A & \xrightarrow{f} & B \end{array}$$

and homomorphism $\overline{f} : G(\widetilde{B}|B) \to G(\widetilde{A}|A)$ which satisfies following conditions

$$\widetilde{f}(g \cdot a) = \overline{f}(g) \cdot \widetilde{f}(a).$$

A homomorphism $\overline{f}$ is called *Homomorphism of fundamental groups with respect to $A \to \widetilde{A}$, $B \to \widetilde{B}$ coverings*.

Example 3.3. *Homomorphism of fundamental groups of noncommutative torus*. Let $A = A_\theta = \mathbb{C}[u, v]$, $B = A_{\theta mn} = \mathbb{C}[u', v']$ be C^* algebras and $f : A \to B$ *- homomorphism defined by following way:

$$u \mapsto u'^m,$$

$$v \mapsto v'^n.$$

Let $A \to \widetilde{A}$, $B \to \widetilde{B}$ coverings defined by following way:

$$\widetilde{A} = A_{\theta/m'n'} = \mathbb{C}[\widetilde{u}, \widetilde{v}],\ \widetilde{B} = A_{\theta/mm'nn'} = \mathbb{C}[\widetilde{u'}, \widetilde{v'}],$$

$$u \mapsto \widetilde{u}^{m'},\ v \mapsto \widetilde{v}^{n'},\ u' \mapsto \widetilde{u'}^{mm'},\ v' \mapsto \widetilde{v}^{nn'}.$$

It is clear that $G(\widetilde{A}, A) \approx \mathbb{Z}_{m'} \times \mathbb{Z}_{n'}$, $G(\widetilde{B}, B) \approx \mathbb{Z}_{mm'} \times \mathbb{Z}_{nn'}$. Homomorphism $\widetilde{f} : \widetilde{A} \to \widetilde{B}$ is defined as:

$$\widetilde{u} \mapsto \widetilde{u'}^{m'}, \widetilde{v} \mapsto \widetilde{v'}^{n'}.$$

These homomorphisms satisfy conditions of definition 3.2. Direct checking shows that Homomorphism of fundamental groups with respect to above coverings is natural surjective homomorphism:

$$\overline{f} : \mathbb{Z}_{mm'} \times \mathbb{Z}_{nn'} \to \mathbb{Z}_{m'} \times \mathbb{Z}_{n'}.$$

4 Abelian coverings

4.1 Abelian fundamental group

Calculation of Galois groups can be very difficult problem. Often difficult problem is replaced by simplified one. For example *class field theory* [18] is a powerful tool calculation of Abelian Galois groups of field extension. Described in 1.2 construction requires analogue of pointed space. However as it is noted in 1.6 if all Galois groups are Abelian then one do not need analogue of pointed space.

Definition 4.1. A finite covering is called an *Abelian covering* if its Galois group is Abelian group.

Definition 4.2. Let A be C^* - algebra. Let us consider category of Abelian coverings of A and covering morphisms. This category induces diagram of Abelian groups and surjective homomorphisms. Inverse limit of this diagram is named an *Abelian fundamental group* of A.

4.2 Canonical constructions of cyclic coverings

Cyclic covering is covering with cyclic Galois groups. Some of cyclic coverings can be obtained by canonicals construction which is described below.

Definition 4.3. Let A be C^* - algebra which has such unitary $u \in U(A)$ that:

1. u^n is not homotopic to 1_A $\forall n \in \mathbb{N}$.
2. $u \neq v^n$; $\forall v \in A$, $\forall\ n \in \mathbb{N}$ & $n > 1$.

Canonical cyclic covering of degree n is such C^* - algebras inclusion $A \subset B$ that B as C^* algebra is generated by single unitary element $v \in B$ that $v^n = u$ ($n > 1$ & $n \in \mathbb{N}$). Element v is called *generator of canonical cyclic covering*.

Example 4.4. *Cyclic construction provided by fundamental group* Let X, x_0 be pointed space and $f : (S^1, s_0) \to (X, x_0)$ generates cyclic element $[f] \in \pi_1(X, x_0)$ with infinite order. Also let $g : X \to S^1$ be such continuous map that gf is homotopic to 1_{S^1}. Map g generates unitary element $u \in U(C(X))$ because $S^1 \approx U(1)$. Let A be C^* - algebra and $u \in U(A)$ is such unitary that $[u] \in K^1(A)$ is nontrivial element that $n[u] \neq 0\&[u] \neq kx$ $\forall n \in \mathbb{N}$ $\forall m \in \mathbb{N}\&m > 1$ $\forall x \in K(A)$. Element u comply conditions of definition 4.3.

Example 4.5. *Cyclic construction provided by K theory* Let A be C^* - algebra and $u \in U(A)$ is such unitary that $[u] \in K^1(A)$ is nontrivial element that $n[u] \neq 0\&[u] \neq kx$ $\forall n \in \mathbb{N}$ $\forall m \in \mathbb{N}\&m > 1$ $\forall x \in K(A)$. Element u comply conditions of definition 4.3.

Example 4.6. *Canonical cyclic construction of noncommutative torus.* Let A_θ be algebra of noncommutative torus (See definition 1.9) and $u, v \in U(A_\theta)$ unitary generators. Then $K^1(A_\theta) \approx \mathbb{Z}^2$ and $[u], [v] \in K^1(A_\theta)$ are generators of $K^1(A_\theta)$. Elements u and v comply to definition 4.3.

Lemma 4.7. *Let $A \subset B$ be canonical cyclic covering of degree n. Then $A \subset B$ is finitely listed covering (See definition 1.8) and its covering group is isomorphic to $\mathbb{Z}_n$.*

Proof. Let $v \in B$ be generator of cyclic covering and $v^n = u \in B$. Let us consider *-automorphism α of B which is defined by following rule:

$$v \mapsto ve^{2\pi i/n},$$

$$v^* \mapsto v^* e^{-2\pi i/n}.$$

It is clear that $\alpha^n = Id_A$. So α is generator of cyclic group $\mathbb{Z}_n$ It is clear that $A = B^{\mathbb{Z}_n}$. Let $C(u) \in A$ be C^* algebra which is generated by u. According to condition 1 of definition 4.3 u^n is not homotopic 1_A $\forall n \in \mathbb{N}$. If S^1 is circle then we have canonical isomorphism $C(u) \approx C(S^1)$. Also we have isomorphism $C(u) \approx C(S^1)$. Inclusion $A \subset B$ generates injective *-homomorphism $C(u) \to C(v)$ which is defined as

$$u \mapsto v^n. \tag{19}$$

According to direct calculation *-homomorphism (19) induces n listed covering

$$S^1_n \to S^1 \tag{20}$$

where S^1_n is a circle. We use S^1_n notation instead S^1 for avoiding ambiguity. Here and later $S^1_n \to S^1$ is n - listed covering of circle S^1 by circle S^1_n. Element $\alpha \in \mathbb{Z}_n$ acts on S^1_n as

$$\phi \mapsto \phi + 2\pi i/n.$$

Since S^1_n is compact it is such finite set $U_i \subset S^1_n$ ($i = 1, ..., N$) of open subsets that

$$\bigcup_{i=1,...,N} U_i = S^1_n,$$

$$gU_i \bigcap U_i = \emptyset, \ i = 1, ...N, \forall g \in \mathbb{Z}_n \tag{21}$$

$U_i \subset S^1$ that $\bigcup_i U_i = S^1$, $gU_i \bigcap U_i = \emptyset$ i = 1,...N, $\forall g \in \mathbb{Z}_n$. Also there are such nonnegative real functions $e_i \in C(S^1)$ that

1. $e_i(x) = 0 \ \forall x \notin U_i$;
2. $\sum_{i=1,...,N} e_i = 1_{C(S^1_n)}$

According to (21) we have $e_i g e_i = 0$; $i = 1, ...N$; $\forall g \in \mathbb{Z}_2$. Now we if one set $a_i = b_i = \sqrt{e_i}$ then one have

$$\sum_{i=1,...,N} a_i b_i = 0;$$

$$\sum_{i=1,...,N} a_i g b_i = 0; \ \forall g \in \mathbb{Z}_n \& \ g \neq e.$$

To above equations are in fact necessary conditions of definition 1.5 □

Example 4.8. *Canonical cyclic construction of circle.* Algebra $C(S^1)$ is canonically represented in Hilbert space $L^2(S^1)$ of square integrated functions. Let $S^1_n \to S^1$ n - listed covering. Then $C(S^1_n)$ is canonically represented in $L^2(S^1_n)$. Also it is canonical injection $L^2(S^1) \to L^2(S^1_n)$. Image of this inclusion is generated by such functions $f \in L^2(S^1_n)$ that $f(e^{i\phi}) = f(e^{i(\phi+2\pi/n)})$. Let v be canonical generator of $C(v) = C(S^1_n)$. Then we can represent $L^2(S^1_n)$ as following direct sum of Hilbert spaces:

$$L^2(S^1_n) = \bigoplus_{i=0,\dots,n-1} v^i L^2(S^1);$$

This direct sum provides following matrix representation of v and its adjoint:

$$v = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & u \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & & & & & \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}; \ v^* = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & & & & & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ u^* & 0 & 0 & \dots & 0 & 0 \end{pmatrix}. \tag{22}$$

Following equations can be proofed directly.

$$v^n = \begin{pmatrix} u & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & & & & & \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}; \ v^{*n} = \begin{pmatrix} u^* & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & & & & & \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix};$$

$$v^*v = v^*v = 1_{L^2(S^1)}.$$

So we can define action of $C(S^1)(v)$ as on Hilbert space $L^2(S^1_n) = \bigoplus_{i=0,\dots,n-1} v^i L^2(S^1)$ by following way.

$$ah = \begin{pmatrix} a & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & & & & & \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} h_0 \\ \dots \\ h_n \end{pmatrix} \ \forall a \in C(S^1), \ h_0, \dots, h_n \in L^2(S^1).$$

Action of v and v^* is represented by matrixes (22).

Example 4.9. *First general canonical cyclic construction.* This example is generalization of above example. Let A be C^* - algebra with unitary element $u \in U(A)$ which satisfies conditions of definition 4.3. Let us select any faithful representation of A on Hilbert space

H. Also let us define action of A on H^n by following rule.

$$a\begin{pmatrix}h_0\\ ...\\ h_n\end{pmatrix}=\begin{pmatrix}a & 0 & 0 & ... & 0 & 0\\ 0 & 1 & 0 & ... & 0 & 0\\ 0 & 0 & 1 & ... & 0 & 0\\ 0 & 0 & 0 & ... & 0 & 0\\ ... & & & & & \\ 0 & 0 & 0 & ... & 0 & 1\end{pmatrix}\begin{pmatrix}h_0\\ ...\\ h_n\end{pmatrix}\ \forall a\in A,\ h_0,...,h_n\in H.$$

Let us define two adjoint operators on $B(H^n)$ by following way

$$v\begin{pmatrix}h_0\\ ...\\ h_n\end{pmatrix}=\begin{pmatrix}0 & 0 & 0 & ... & 0 & u\\ 1 & 0 & 0 & ... & 0 & 0\\ 0 & 1 & 0 & ... & 0 & 0\\ 0 & 0 & 1 & ... & 0 & 0\\ ... & & & & & \\ 0 & 0 & 0 & ... & 1 & 0\end{pmatrix}\begin{pmatrix}h_0\\ ...\\ h_n\end{pmatrix};\ v^*\begin{pmatrix}h_0\\ ...\\ h_n\end{pmatrix}=\begin{pmatrix}0 & 1 & 0 & ... & 0 & 0\\ 0 & 0 & 1 & ... & 0 & 0\\ 0 & 0 & 0 & ... & 0 & 0\\ 0 & 0 & 0 & ... & 0 & 0\\ ... & & & & & \\ 0 & 0 & 0 & ... & 0 & 1\\ u^* & 0 & 0 & ... & 0 & 0\end{pmatrix}\begin{pmatrix}h_0\\ ...\\ h_n\end{pmatrix}.$$

In this case $v^n = u$ as operator on H^n.

Definition 4.10. Described in example construction of canonical cyclic covering 4.9 is called a *frequency domain construction*.

Remark 4.11. *Coverings of noncommutative torus once again*. Coverings of noncommutative torus were already considered in example 1.9. Defined in example 4.9 construction provides alternative search of these coverings. First of all according to example 4.6 generators $u, v \in U(A_\theta$ comply conditions of definition 4.3. Using this property on can construct coverings by canonical cyclic construction.

Example 4.12. *Spectral canonical cyclic construction of circle* If functional space $L^2(S^1)$ does not depend on values on zero measures set then $L^2(S^1) \approx L^2([0,2\pi]) \approx L^2([0,2\pi))$. Similarly if we consider spaces of f all bounded complex valued Borel-measurable functions then we have $B_\infty(S^1) \approx B_\infty([0,2\pi]) \approx B_\infty([0,2\pi))$ Let us parameterize $L^2(S^1_n)$ by real parameter $\phi \in [0,2\pi)$. Let $H_k \subset L^2(S^1_n)$ $(k = 0,...,n-1)$. is subset of functions which are supported on $2\pi k/n \le \phi < 2\pi(k+1)/n$. We have following decomposition

$$L^2(S^1_n) = \bigoplus_{k=0,...,n-1} H_k.$$

Similarly we have decomposition

$$B_\infty(S^1_n) = \bigoplus_{k=0,...,n-1} B_{\infty,k}$$

where $B_{\infty,k}$ is supported on $2\pi k/n \le \phi < 2\pi(k+1)/n$. Otherwise we have following natural isomorphisms:

$$H_k \approx L^2(S^1);\ B_{\infty,k} \approx B_\infty(S^1)\ k = 0,...,n-1;$$

Every element $a \in B_\infty(S^1)$ acts on diagonally acts on direct sum $\bigoplus_{k=0,...,n-1} B_{\infty,k}$.

$$a \begin{pmatrix} h_0 \\ ... \\ h_n \end{pmatrix} = \begin{pmatrix} a & 0 & 0 & ... & 0 \\ 0 & a & 0 & ... & 0 \\ ... \\ 0 & 0 & 0 & ... & a \end{pmatrix} \begin{pmatrix} h_0 \\ ... \\ h_n \end{pmatrix} \quad \forall a \in B_\infty(S^1)$$

Let us define functions $f_0, ..., f_{n-1} \in B_\infty(S^1)$ by following equations:

$$\phi \mapsto e^{2\pi i(\phi+k)/n}. \tag{23}$$

Generator $v \in C(S^1_n)$ acts on direct sum as

$$\begin{pmatrix} f_0 & 0 & 0 & ... & 0 \\ 0 & f_1 & 0 & ... & 0 \\ ... \\ 0 & 0 & 0 & ... & f_{n-1} \end{pmatrix}$$

It is clear that v satisfies all conditions of definition 4.3

Example 4.13. *Second (spectral) general canonical cyclic construction* Here generalization of example is considered. Let A be C^* - algebra with unitary element $u \in U(A)$ which satisfies conditions of definition 4.3. Let us select any faithful representation of A on Hilbert space H. Also let us define action of A on H^n by following rule.

$$a \begin{pmatrix} h_0 \\ ... \\ h_{n-1} \end{pmatrix} = \begin{pmatrix} a & 0 & 0 & ... & 0 & 0 \\ 0 & a & 0 & ... & 0 & 0 \\ 0 & 0 & a & ... & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & 0 \\ ... \\ 0 & 0 & 0 & ... & 0 & a \end{pmatrix} \begin{pmatrix} h_0 \\ ... \\ h_{n-1} \end{pmatrix} \forall a \in A, \ h_0, ..., h_n \in H.$$

Since u is normal then then according to spectral theorem [10] we have following natural unital *-homomorphism:

$$Second(spectral)generalcanonicalcyclicconstruction B_\infty(\sigma(u)) \to B(L^2(S^1)), u \mapsto f(u). \tag{24}$$

Following formula is spectral representation of u.

$$u = \int_{\sigma(u)} zEdz$$

where $\sigma(u)$ is spectrum [10] of u. Since u is unitary then $\sigma(u) \subset \{z \in \mathbb{C}, |z| = 1\}$ So we can define action of functions $f_0, ..., f_{n-1} \in B_\infty(\sigma(u))$ defined in 23. Let us define new operator $v \in B(H^n)$ by following rule.

$$v \begin{pmatrix} h_0 \\ ... \\ h_{n-1} \end{pmatrix} = \begin{pmatrix} f_0 & 0 & 0 & ... & 0 & 0 \\ 0 & f_1 & 0 & ... & 0 & 0 \\ 0 & 0 & f_2 & ... & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & 0 \\ ... \\ 0 & 0 & 0 & ... & 0 & f_{n-1} \end{pmatrix} \begin{pmatrix} h_0 \\ ... \\ h_{n-1} \end{pmatrix} \forall a \in A, \ h_0, ..., h_n \in H.$$

Operators $f_0, ... f_{n-1}$ have following spectral representation

$$f_k = \int_{\sigma(u)} f_k(z) E dz; \; k = 0, ..., n-1.$$

Operator v satisfies all conditions of definition 4.3. However one shall not need usage of H^n. One can use $v \in B(H)$ defined by following way:

$$v = \int_{\sigma(u)} f_0(z) E dz; \tag{25}$$

Definition 4.14. Described in example 4.13 is called *time domain construction* of canonical cyclic covering.

Remark 4.15. Time domain construction of provides interesting implementation of coverings. If $\widetilde{X} \to X$ is a covering that $C(\widetilde{X})$ can be obtained by addition to $C(X)$ discontinuous functions which belong to $B_\infty(X)$.

Example 4.16. *Cyclic construction with two generators*. Let X be topological space $\pi_1(X) \approx \mathbb{Z} \times \mathbb{Z}$ and $a_1, a_2 \in \pi_1(X)$ are generators of $\pi_1(X) \approx \mathbb{Z} \times \mathbb{Z}$. According to [11] it is such finitely listed covering $f : \widetilde{X} \to X$ that $\pi_1(f) : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}_m \times \mathbb{Z}_n$ surjective group homomorphism. Corresponding $*$- homomorphism $C(f) : C(X) \to C(\widetilde{X})$ can be constructed be canonical construction of cyclic coverings. First of all suppose that a_1, a_2 are represented by $f_1, f_2 : (S, s_0) \to (X, x_0)$. Then let $g_1, g_2 : X \to S^1$ be such continuous functions that both $g_1 f_1$ and $g_2 f_2$ are homotopic to Id_{S^1}. Let $u_1, u_2 \in U(C(X))$ unitary elements which correspond to g_1, g_2 respectively. Then there we can consider following inclusions:

$$C(X) \to C(X)(v_1) \to C(X)(v_1)(v_2) \approx C(\tilde{X})$$

where $v_1^m = u_1, v_1^n = u_2$.

Example 4.17. Let us suppose that elements v_1, v_2 are constructed by time domain formula (25). In this case either $C(X)$ and $C(\tilde{X})$ act on single Hilbert space H. Let us consider extension B of $C(X)$ generated by v_1', v_2' and

$$v_1'^m = u_1, \; v_2' = u_2. \tag{26}$$

Algebra B is represented in $H \otimes \mathbb{C}^q$ Hilbert space. Subalgebra $C(X)$ diagonally acts on H. Elements v_1', v_2' are defined as operators on $H \otimes \mathbb{C}^q$ by following formulae:

$$v_1' = v_1 \otimes V_1, \; v_2' = v_2 \otimes V_2$$

where $V_1, V_2 \in \mathbb{M}_q(\mathbb{C})$ and

$$V_1^q = V_2^q = 1_{\mathbb{C}^n}; \; V_1 V_2 = e^{2\pi i p/q} V_2 V_1.$$

Conditions can be proved directly. Algebra A is not commutative because

$$v_1' v_2' = e^{2\pi i p/q} v_2' v_1'.$$

So we have obtained new sample of noncommutative covering of commutative C^* - algebra.

Remark 4.18. Noncommutative coverings of commutative torus (See example 1.10) can be constructed as well as in example 4.17.

Example 4.19. *Coverings of noncommutative 3D Sphere* Algebra of complex functions of 3D sphere could be generated by four real valued functions x_1,...,x_4 those satisfy to following equations:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1. \tag{27}$$

If we introduce complex valued functions $\alpha = x_1 + ix_2$, $\beta = x_3 + ix_4$ then when we can replace (27) by the following equation:

$$\alpha\alpha^* + \beta\beta^* = 1. \tag{28}$$

Very interesting involutive noncommutative algebra is considered in [19]. It is generated by two elements α, β and satisfies to following relations.

$$\alpha^*\alpha + \beta^*\beta = 1;\ \alpha\alpha^* + q^2\beta\beta^* = 1;\ \alpha\beta - q\beta\alpha = 0;\ \alpha\beta^* - q\beta^*\alpha = 0;\ \beta^*\beta = \beta\beta^*. \tag{29}$$

where q is a real number and $0 < q \leq 1$.
By $C(SU_q(2))$ denote C^* which satisfy above equation. It is clear that if we suppose that $q = 1$ then this algebra is commutative and it satisfies to relations (27). If $q \approx 1$ then algebra $C(SU_q(2))$ could be considered as noncommutative approximation of algebra $C(S^3)$ of continuous complex valued functions on 3D sphere. $C(SU_q(2))$ admits the structure of spectral triple[17]. It is well known that 3D sphere is simply connected. So if $q = 1$ then $C(SU_q(2))$ no nontrivial finite coverings. However if $q = 1$ then it is such unitary element $u \in U(C(SU_q(2))$ than $[u] \in K^1(C(SU_q(2))$ is not trivial and has infinite period. According to example 4.5 element u comply conditions of definition 4.3. So one can construct construct cyclic covering $C(SU_q(2)) \to B$ where be is generated over a by such element v that $v^n = u$.

Remark 4.20. If $q \approx 1$ then algebraic properties of $C(SU_q(2))$ are very close to algebraic properties of commutative algebra $C(S^3)$. However these algebras are principally different. First one does not have nontrivial coverings but second one has them. Perhaps this fact is relevant to structure of the Universe. In some models space of the Universe is $C(S^3)$. Since $C(SU_q(2))$ is close $C(S^3)$ it is reasonably suppose that Universe space correspond to algebra $C(SU_q(2))$. Since former algebra has nontrivial coverings this fact can occur new cosmological properties.

4.3 Analogy with Kummer extensions

Here analogue with algebraic field extensions is considered. Let K be a field $\overline{K}$ (resp. K_{sep})is its algebraic (resp. separable) closure and $\mathfrak{G} = G(\overline{K}/K)$ is Galois group. Cyclic extension [18] L of K ($K \subset L \subset \overline{K}$) have such Galois group $\mathfrak{H} \subset \mathfrak{G}$ that $\mathfrak{G}/\mathfrak{H}$ is finite cyclic group. If number of elements of $\mathfrak{G}/\mathfrak{H}$ is equal to $n \in \mathbb{N}$ then $\mathfrak{G}/\mathfrak{H}$ is isomorphic to group of n-th roots of unity in $\mathbb{C}$. This isomorphism may be regarded as character χ of $\mathfrak{G}$

with the kernel $\mathfrak{H}$; such a character which if order of n, will be said *attached to L*. If α is representative in $\mathfrak{G}$ of generator of $\mathfrak{G}/\mathfrak{H}$, there is one and only one character χ of $\mathfrak{G}$ that $\chi(\alpha) = e^{2\pi i/n}$. Conversely if χ any homomorphism from $\mathfrak{G}$ into $\mathbb{C}^{\times}$; it is a character of order n; its kernel is open subgroup $\mathfrak{H} \in \mathfrak{G}$ with cyclic subgroup of order n and subfield $L \subset K_{sep}$ corresponding to $\mathfrak{H}$ is cyclic of degree n over K; we will than say that L is *attached to L*. If K contains distinct n roots of 1; then these make up a cyclic group E of order n, if K is of characteristic $p > 1$, or assumption implies that n is prime to p. Let ψ be an isomorphism of E onto group of n-th roots of 1 in $\mathbb{C}$; this will be determined uniquely if we choose a generator ϵ_1 of E and prescribe $\psi(\epsilon_1) = e^{2\pi i/n}$. Take any $\xi \in K^{\times}_{sep}$, and let x be any one of roots of equation $X^n = \xi$ in $\overline{K}$; when $x \in K^{\times}_{sep}$, and equation $X^n = \xi$ has n distinct roots ϵx with $\epsilon \in E$. In particular, for each $\sigma \in \mathfrak{G}$ x^{σ} must be one of the roots, so that $x^{\sigma}x^{-1}$ is in E. Now put

$$\chi_{n,\xi}(\sigma) = \psi(x^{\sigma}x^{-1}); \tag{30}$$

as $E \in K$, the right-hand side does not change if we replace x by ϵx with $\epsilon \in E$ and is therefore independent of choice of a root x for $X^n = \xi$. For similar reason, we have, for all, $\rho, \sigma \in \mathfrak{G}$;

$$x^{\rho\sigma}x^{-1} = (x^{\rho}x^{-1})^{\sigma}(x^{\sigma}x^{-1}) = (x^{\rho}x^{-1})(x^{\sigma}x^{-1}),$$

and therefore

$$\chi_{n,\xi}(\rho\sigma) = \chi_{n,\xi}(\rho)\chi_{n,\xi}(\sigma),$$

and therefore shows that $\chi_{n,\xi}$ is a character on $\mathfrak{G}$. Take now any $\eta \in K^{\times}$, and call y a root of $X^n = \eta$; then xy is root of $X^n = \xi\eta$, and we have for all $\sigma \in \mathfrak{G}$):

$$(xy)^{\sigma}(xy)^{-1} = (x^{\sigma}x^{-1})(y^{\sigma}y^{-1})$$

end therefore

$$\chi_{n,\zeta\eta} = \chi_{n,\zeta}\chi_{n,\eta},$$

which shows that $\xi \mapsto \chi_{n,\xi}$ is a morphism of $K^{\times}$ into group of characters of $\mathfrak{G}$. It is obvious that $\chi_{n,\xi}$ is trivial if $X^n = \xi$ has one root, hence all its roots, in K, i.e. if $\xi \in (K^{\times})^n$; in other words, $(K^{\times})^n$ is kernel of $\xi \mapsto \chi_{n,\xi}$. It would be easy to show that the image of $K^{\times}$ under that morphism consists of all the characters of $\mathfrak{G}$ whose order divides n, but this will not be needed. Let us generalize this construction. Let A be C^* - algebra, $U(A)$ group of its unitary elements, $U_0(A) \in U(A)$ subgroup homotopic to unity elements, $[U(A)] = U(A)/U_0(A)$ factorgroup. By $[[U(A)], [U(A)]]$ denote commutator of $[U(A)]$, by $[U(A)]_{ab}$ denote factorgroup $[U(A)]/[[U(A)], [U(A)]]$. Let $\mathrm{Tors}([U(A)]_{ab}) \in [U(A)]_{ab}$ be subgroup of elements which have finite period. Factorgroup $[U(A)]_{ab\ free} = [U(A)]_{ab}/\mathrm{Tors}([U(A)]_{ab})$ is free Abelian group. Let $u_1, ..., u_p \in U(A)$ such unitary elements that:

1. u_i satisfy conditions of definition 4.3 for $i = 1, ..., p$.

2. Classes $\overline{u}_i \in [U(A)]_{ab\ free}$ are linearly independent.

According to example 4.13 and/or example 4.9 $\forall \in \mathbb{N}$ one can Abelian construct finite covering $A \to B$ that there are such unitary elements $v_1, ..., v_k \in B$ that $v_k^n = u_k$. By $\mathfrak{G}$ denote Abelian Galois group of this covering. This group can be represented as following direct sum.

$$\mathfrak{G} = \bigoplus_{j=1,...,k} \mathfrak{G}_j;\ \mathfrak{G}_j \approx \mathbb{Z}_n \forall j\ (1 \le j \le k);$$

And summand $\mathfrak{G}_j$ is generated by automorphism $\sigma \in \mathfrak{G}$ which acts on B by following way:

$$v_j \mapsto e^{2\pi i/n} v_j;$$

$$v_l \mapsto v_l;\ l \neq j.$$

Now we can define character $\xi_{n,\overline{u}_j}$ on defined as

$$\xi_{n,\overline{u}_j}(\sigma) = \sigma(v_j) v_j^{-1};$$

This equation can be regarded as analogue of equation 30.

5 Generalization of infinite covering

According to section 1 C^* it is following mapping:

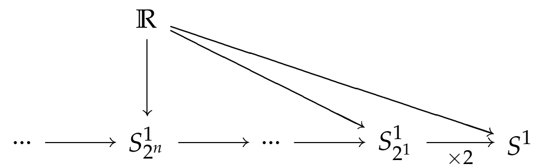
TOPOLOGY	ALGEBRA
Locally compact space	C^* - algebra
Compact space	Unital C^* - algebra
Continuous map	*-homomorpfism

This map excludes generalization of infinitely listed coverings by following reasons. Let X be compact Hausdorff space and $p : Y \to X$ be infinitely listed covering, then Y is not compact, $C_0(X)$ is unital, but $C_0(Y)$ is not unital. Homomorphism $C(p) : C_0(X) \to C_0(Y)$ which correspond to p does not exist.So one should generalize notion of *- homomorphism for generalization of infinitely listed coverings .

5.1 Noncommutative generalization of $\mathbb{R} \to S^1$ covering

5.1.1 $\mathbb{R} \to S^1$ covering

$\mathbb{R} \to S^1$ *covering*. Let $p : \mathbb{R} \to S^1$ well known infinitely listed covering [11]. No function $f \in C_0(\mathbb{R})$, $f \neq 0$ can be obtained from function $g \in C_0(S^1)$. However it is another alternative construction which is modification universal object. Let us consider category of all coverings of circle.



We assume that all coverings of above coverings are two listed coverings. Symbol $S^1_{2^n} \approx S^1$ means that that covering degree of initial circle S^1 is equal to 2^n. It is natural bijection between $C(S^1)$ and $\{f :\in C[-2^n\pi, 2^n\pi], f(-2^n\pi) = f(2^n\pi)]\}$. Any $f \in C_0(\mathbb{R})$ can be regarded as limit of functions supported on intervals $[-2^n\pi, 2^n\pi]$, $n \to \infty$. More precisely for $f \in C_0(\mathbb{R})$ it is following sequence $f_{2^n} \in C_0(\mathbb{R})$:

$$f_{2^n}(x) = \begin{cases} f(x) - f(2^n\pi) + (x - 2^n\pi)\frac{f(x+2^n\pi)-f(x-2^n\pi)}{2^{n+1}\pi}; \ x \in [-2^n\pi, 2^n\pi] \\ 0; \ x \notin [-2^n\pi, 2^n\pi] \end{cases} \tag{31}$$

It is evident that sequence f_{2^n} is norm convergent to f. Now let us note that

$$L^2(\mathbb{R}) = \bigoplus_{k\in\mathbb{Z}} L^2([2\pi k, 2\pi(k+1)]); \tag{32}$$

where $\oplus$ means Hilbert direct sum. Otherwise f_{2^n} can be naturally identified with element $\overline{f}_{2^n} \in S^1_{2^n}$ It is clear that there are following natural isomorphisms of Hilbert spaces

$$L^2([2\pi k, 2\pi(k+1)]) \approx L^2([2\pi n, 2\pi(n+1)]) \ \forall k, n \in \mathbb{Z}.$$

Let us also define action $\overline{f}_{2^n} \in S^1_{2^n}$ on $L^2(\mathbb{R})$ as action of f_{2^n}. It is clear that $\overline{f}_{2^n}$ trivially acts on $L^2([-2\pi k, 2\pi(k+1)])$ $(n < k \bigvee n > k+1)$.

5.1.2 Generalization of $\mathbb{R} \to S^1$ covering

Algebra $C(S^1)$ satisfies conditions of definition 4.3. The $\mathbb{R} \to S^1$ covering can be generalized on any C^* algebra which satisfies conditions of definition 4.3. Let A and $u \in U(A)$ be such algebra and its unitary element which satisfy to conditions of definition 4.3. Algebra A faithfully acts on Hilbert space H. From lemma 4.7 it follows that there exists following sequence of finitely coverings:

$$... \longleftarrow A[v_{2^n}, v^*_{2^n}] \longleftarrow ... \longleftarrow A[v_2, v^*_2] \longleftarrow A$$

where $v^{2n} = u$.
Above diagram is noncommittal analogue of diagram considered at 5.1.1. Indeed $C(S^1) \approx C(u)$ and $C(S^1_{2^n}) \approx C(v_{2n})$. Let us define action of $C_0(\mathbb{R})$ on Hilbert sum $\widetilde{H} = \bigoplus_{n\in\mathbb{Z}} H_n$. First of all note that $C(v_{2^k})$ acts on $\bigoplus_{-2^k \leq n < 2^k - 1} H_n$. Suppose that $C(v_{2^k})$ acts trivially on H_n if $n < -2^k \bigvee n \geq 2^k$. This action can be continued whole sum $\widetilde{H}$. Let $f \in C(\mathbb{R})$ any function and sequence f_n is defined by equation 31. On can define functions $\overline{f}_{2^n} \in C(S^1_{2^n}) = C(v_{2^n})$. So l f_{2^n} $\forall n \in \mathbb{N}$ defines bounded operator $B(f_{2^n}) \in B(\widetilde{H})$. Sequence

$B(f_{2^n})$ is norm convergent. By $B(f)$ denote its limit. Denote $B(a) \in B(\widetilde{H})$ $a \in A$ bounded operator which acts on every component of Hilbert sum as well as a acts on H.

Definition 5.1. Let $B \in B(\widetilde{H})$ be norm completion of algebra generated by elements $B(a)B(f)$ and $B(f)B(a)$ where $a \in A$ and $f \in C(\mathbb{R})$. This algebra is called *Noncommutative generalization of* $\mathbb{R} \to S^1$ *covering*.

Remark 5.2. This construction can be generalized. Suppose that there are two elements $u_1, u_2 \in A$ which satisfy conditions 4.3. We can define two actions of $C(\mathbb{R})$ on $\widetilde{H}$. Let us distinguish these actions for clarity. Action of $C(\mathbb{R})_1$, $C(\mathbb{R})_2$ is constructed by usage of elements u_1 and u_2 respectively. Norm completion of algebra generated by $B(f_1)B(f_2)B(a)$,$B(f_2)B(f_1)B(a)$, $B(f_1)B(a)B(f_2)$, $B(f_2)B(a)B(f_1)$, $B(a)B(f_1)B(f_2)$, $B(a)B(f_2)B(f_1)$ where $a \in A$, $f_1 \in C(\mathbb{R})_1$ $f_2 \in C(\mathbb{R})_2$, can be regarded as generalization of covering of torus be plane. Similarly covering of n torus by R^n can be generalized.

Example 5.3. *Infinite covering of noncommutative torus.* Algebra A_θ of noncommittal torus has two unitary elements u, v which satisfy conditions of definition 4.3. So infinite generalization of covering by plane can be constructed. Since v satisfies condition 4.3 one can construct such sequence $v_2, v_2^*, ..., v_{2^n}, v_{2^n}^*$ that $v_{2^n}^{2n} = v$, $v_{2^n}^{*2n} = v^*$; $\forall n \in \mathbb{N}$. Elements of this sequence satisfy following conditions:

$$uv_{2^n} = e^{2\pi i(\theta+k)/2^n} v_{2^n} u$$

where $k \in \mathbb{Z}$ is arbitrary integer number. Here we set $k = 0$. In this case $uv_{2^n} = e^{2\pi i\theta/2^n} v_{2^n} u$. Sequence v_{2^n} induces sequence $B(f_{2^n}) \in B(\widetilde{H})$ for all $f \in C_0(\mathbb{R})$. $B(f) \in B(\widetilde{H})$ is norm limit of $B(f_{2^n})$ (See 5.1.2). Operators $B(u)$ and $B(f_{2^n})$ satisfy following condition.

$$B(u)B(f_{2^n}) = e^{2\pi i\theta/2^n} B(f_{2^n})B(u).$$

Since $B(f)$ is norm limit of $B(f_{2^n})$ we have.

$$B(u)B(f) = B(f)B(u).$$

From previous equation it follows that algebra generated by elements $B(u)$ and $B(f)$ $\forall f \in C_0(\mathbb{R})$ is commutative. So its norm completion is also commutative. One can check that this algebra is isomorphic to $C_0(S^1 \times \mathbb{R})$. Generalization of infinite covering of $C_0(S^1 \times \mathbb{R})$ is $C_0(\mathbb{R}^2)$. This generalization coincides with commutative covering. So infinite covering of noncommutative torus is commutative plane.

5.2 Generalization of arbitrary infinite covering

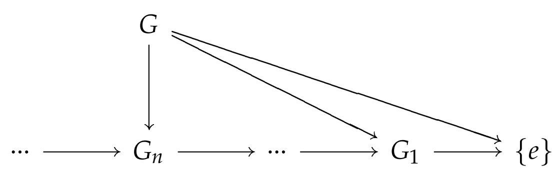
Here we would like construct generalization of arbitrary infinite covering. This construction is analogical to commutative infinite covering. So first of all algebraic construction of commutative infinite covering will be constructed.

5.2.1 Commutative infinite covering from algebraic viewpoint

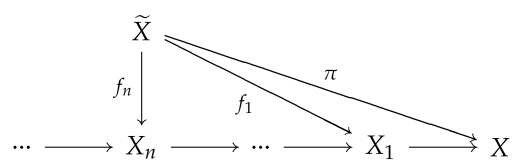
Let (X, x_0) be pointed topological space space, $\pi(\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is infinitely listed covering, $G = G(\widetilde{X}, X)$ is covering group. According to GNS Construction [7] C^* - algebra $C(X)$ has a faithful representation, i.e. $C(X)$ is isometrically isomorphic to C^*-algebra of operators on a Hilbert space H. Here full representation of $C(\widetilde{X})$ on Hilbert sum

$$\widetilde{H} = \oplus_{g \in G(\widetilde{X}, X)} H_g;\ H_g \approx H\ (\forall g \in G(\widetilde{X}, X)) \tag{33}$$

will be constructed. Let $U \in \widetilde{X}$ be connected fundamental domain i.e. U is open, limitation $\pi|_U$ is injective map, and $\pi(U) \in X$ is dense subset. Suppose that $\widetilde{x}_0 \in U$. Group $G(\widetilde{X}, X)$ acts on $\widetilde{X}$. Group G acts on $\widetilde{X}$ and gU is fundamental domain $\forall g \in G$. Denote by A'' bicommutant of C^* - algebra A [7]. Any faithful action of A on Hilbert space induces faithful action of A'' on same Hilbert space. Since $\pi(U)$ is dense in X we have $C(\pi(U))'' = C(X)''$. Set $\widetilde{U} = \pi^{-1}(\pi(U)) = \bigcup_{g \in G} gU$ is dense open subset of $\widetilde{X}$, $C(\widetilde{X})'' \approx C\widetilde{U}'' \approx \bigoplus_{g \in G} C(gU)''$. Otherwise $\bigoplus_{g \in G} C(gU)''$ acts on Hilbertian sum $\widetilde{H} = \bigoplus_{g \in G} H_g$ where $H_g \approx H\ \forall g \in G$. So $C(\widetilde{X})$ have faithful representation on $\widetilde{H} = \bigoplus_{g \in G} H_g$. Action of $\widetilde{a} \in C(\widetilde{X}$ is defined by following way. Element $\widetilde{a}$ is continous function on $\widetilde{X}$. Its limitation $\widetilde{a}|_{gU}, (g \in G)$ is element of $C(gU)$, $\widetilde{a} \in C(gU)''$. So $\widetilde{a}|_{gU}$ acts on H_g. Action of $\widetilde{a}$ on $\widetilde{H} = \bigoplus_{g \in G} H_g$ is componentwise action of $\widetilde{a}|_{gU}$ on $H_g\ \forall g \in G$. Let us consider approximation of this action by actions obtained by finite coverings. Suppose that G can be included into following diagram of surjective group homomorphism:



where G is finite $\forall n \in \mathbb{N}$. Suppose that $\bigcap_{n \in \mathbb{N}} \ker(G \to G_n) = \{e\}$. This diagram induces following of coverings.



where $G(X_n|X) \approx G_n$, maps $p_n : X_n \to X\ \forall n \in \mathbb{N}$ are finite coverings.
Let $g_1, ..., g_k$ be all elements of G. G_n is factorgroup of G. Let us select for all $g_i \in G_n$ such representative $\widetilde{g}_i$ that set $\overline{\bigcup_{i=1,...,m} \widetilde{g}_i U}$ is connected. Set $p(U_n) = \bigcup_{i=1,...,m} \widetilde{g}_i U$ is dense open subset of X_n. Slight modification of previous speculations shows that $C(X_n)$ have full representation on direct sum $\bigoplus H_{\widetilde{g}_i}$ $(i = 1, ..., m)$. Since $\bigoplus H_{\widetilde{g}_i}$ $(i = 1, ..., m) \subset \widetilde{H}$ $C(X_n)$ acts on $\widetilde{H}$. Let us select such fundamental domains $U_i \in X$, $i \in \mathbb{N}$ which correspond to

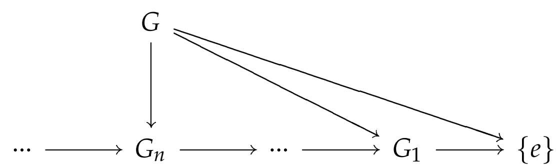
spaces X_i and $U_i \subset U_j$, $(i < j)$. This selection define actions of $C(X_i)$ $i \in \mathbb{N}$ and all these actions are compatible with homomorphisms $C(X_i) \to C(X_j)$. Let $A \in B(\widetilde{H})$ be norm completion of algebra generated by $C(X_i) \in B(\widetilde{H})$. $C(X) \in A$ is such subalgebra that if $a \in C(X)$ that for all $\varepsilon > 0$ number of such spaces H_g that $\|a|_{H_g}\| > \varepsilon$ is finite.

5.2.2 Noncommutative algebraic generalization of infinite covering

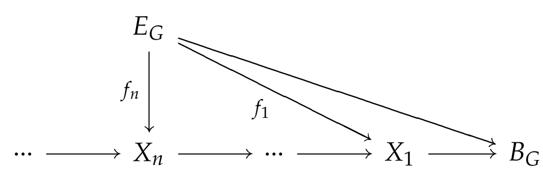
Let A be C^* algebra and

$$\dots \qquad A_n \longleftrightarrow \dots \longleftarrow A_1 \longleftarrow A$$

sequence of finite coverings, and corresponding sequence of covering groups can be included into following diagram:



Also suppose that $\bigcap_{n \in \mathbb{N}} \ker(G \to G_n) = \{e\}$. Milnor's construction [8] provides such infinite covering space B_G that $\pi_1(B_G) \approx G$. Universal covering of this space is usually denoted by $E_G \to B_G$, G acts on E_G and $E_G/G \approx B_G$. This covering induces following diagram: This diagram induces following of coverings:



where $G(X_n, B_G) \approx G_n$, $\forall n \in \mathbb{N}$. Let $U \in E_G$ fundamental domain. For all elements $g_1, ..., g_m \in G_n$ we will select such representatives $\widetilde{g}_i \in G$ that set $\overline{\bigcup_{i=1,..,m} \widetilde{g}_i U_n}$ is connected. In this case $U_n = \bigcup_{g \in G_n} \widetilde{g} U$ is fundamental domain of $X_n \to B_G$ covering.

Let $A \to B(H)$ be GNS representation. Let $\widetilde{H} = \bigoplus_{g \in G} H_g$ is Hilbertian sum. Constructed algebra of infinite coverings subalgebra of $B(\widetilde{H})$. GNS representation is Hilbertian sum of irreducible representations. Irreducible representations of A will be indexed by set Λ i.e. $r_\lambda : A \to B(H_\lambda)$, $(\lambda \in \Lambda)$. Hilbert space of GNS representation is following Hilbert sum $H = \bigoplus_{\lambda \in \Lambda} H_\lambda$. According to [7] . Let $r : A_n \to B(H')$ any irreducible representation. According to [10] there exist irreducible limitation $r' : A \to B(K)$ where $K \subset H'$. A is hereditary subalgebra of A_n because A_n is finitely generated projective module. According to [10] $K = H'$ or it is such unique $\lambda \in Lambda$ that $H' = H_\lambda$ For any representation $r_\lambda : A \to H$ unique extension $r'_\lambda : A_n \to H$ will be fixed. Let $\phi_\lambda : A_n \to \mathbb{C}$

be positive functional which defines representation r'_λ. By $\phi_{\lambda g}$ denote following positive functional

$$\phi_{\lambda g}(a) = \phi_\lambda(ga);\ \forall a \in A_n, g \in G_n.$$

Representation defined by $\phi_{\lambda g}$ has same limitation on A as ϕ_λ one. By H_g denote Hilbertian sum of spaces of representations $\phi_{\lambda g}$, $\forall \lambda \in \Lambda$. So Hilbert space of GNS representation of A_n is Hilbertian is a direct sum $H_n = \bigoplus_{g \in G_n} H_g$. For all covering group $G_n = \{g_{n,1}, ..., g_{n,m}\}$ we will select such representatives $\widetilde{g}_{n,1}, ..., \widetilde{g}_{n,m} \in G$ that $U_n = \bigcup_{i=1,..,m} \widetilde{g}_{n,i} U$ is connected fundamental domain. Fundamental domains are selected by such way that if $i < j$ then $U_i \subset U_j$. Selection of these representatives enable us define action of A_n on Hilbertian sum $\widetilde{H} = \bigoplus_{g \in G} H_g$. Actions of algebras A_n are compatible with finite coverings $A_i \to A_j$. Let $B \in B(\widetilde{H})$ be norm completion of algebra generated by all elements $a \in A_n$, $n \in \mathbb{N}$. Let $\widetilde{A} \subset B$ be such subalgebra that for all $\varepsilon > 0$ number of elements $g \in G$ which satisfy condition $\|a|_{H_g}\| > \varepsilon$ is finite.

Definition 5.4. In this sitation algebra $\widetilde{A}$ is named *generalization of infinite covering*.

6 Generalization of Hurewicz homomorphism

Notion of Hurewicz homomorphism was initially appeared in algebraic topology and then generalized in several directions. This chapter is devoted to generalization related to theory of C^* - algebras. First of all let us remind some notions of algebraic topology [11]. Let X be topological space, $x_0 \in X$ is base point, $\pi_n(X, x_0)$, $H_n(X)$ are n-th ($n \in \mathbb{N}$) are n-th homotopy group and singular homology group respectively. Then $\forall n \in \mathbb{N}$ there is natural homomorphism $\phi_n : \pi_n(X, x_0) \to H_n(X)$. This homomorphism is named *Hurewicz homomorphism*. Pair (X, x_0) is named pointed space. if $n = 1$ then homomorphism is defined by following way: Let S^1 be a circle then $H_1(S^1) \equiv \mathbb{Z}$. Let $c \in H_1(S^1)$ be generator of $H_1(S^1)$. Then Hurewicz homomorphism is defined by following expression: Here we consider generalization of ϕ_1 only and we shall replace ϕ_1 by ϕ for simplicity. For generalization of Hurewicz homomorphism we need answer following questions:

1. What is analogue of $H_1(X)$?
2. What is analogue of pointed space (X, x_0)?
3. What is analogue of $\pi_1(X, x_0)$?
4. What is analogue of Hurewicz homomorphism?

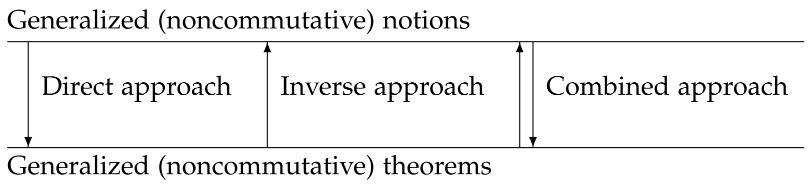
There is a set of versions of answers which depend on context. Analogue of $H_1(X)$ for Hurewicz theorem can be different from analogue of $H_1(X)$ for other problems. There are three approaches for noncommutative generalization of classical (commutative) geometrical results.

1. *Direct (Deductive)* From analogues of definitions to analogues of theorems;

2. *Inverse* From analogues of theorems to analogues of definitions;

3. *Combined* Simultaneous development of analogues of definitions and theorems.

These approaches are schematically represented below:



For example classical notion of Hurewicz homomorphism is based on notion of fundamental group. However noncommutative generalization of fundamental group can be based on noncommutative generalization of Hurewicz homomorphism. Combined point of view implies simultaneous generalization of both fundamental group and Hurewicz homomorphism.

6.1 Generalization of homology group H_1

Equation **??** can be used as definition of Hurewicz homomorphism. This equation can be generalized for any covariant homotopy invariant functor P which satisfies following condition $P(S^1) \approx \mathbb{Z}$. This generalization is defined by following way:

$$\phi([f]) = P(f)(c) \ \forall([f] \in [S^1, s_0, X, x_0] = \pi_1(X, x_0). \tag{34}$$

This observation provides following requirement for noncommutative generalization of H_1. Generalization of H_1 should be such contravariant homotopy invariant functor P from category of C^*- algebras to category of Abelian groups that $P(C(S^1)) \approx \mathbb{Z}$.

Example 6.1. Let K^1 be functor of K - homology. Then $K^1(C(S^1)) = \mathbb{Z}$. In this article the K^1 as generalization of H_1 is being considered.

6.2 Generalization of pointed space

Let us remind some facts from commutative topology.

1. If $X = \coprod_i X_i$ is disjoint union and all X_i are connected then all algebras $C(X_i)$ are simple and $C(X) = \bigoplus_i C(X_i)$

2. If C^* - algebra $A = \bigoplus_i A_i$ then i - th connected component is associated to A_i (A_i is simple algebra.

3. If A is unital then $1_A = \sum_i 1_{A_i}$ and 1_{A_i} is selfadjoint idempotent of A.

4. Any point $x_0 \in X$ defines homomorphism $H_0(\{x_0\}) \to H_0(X)$ and generator of $h \in H_0(X)$. If $X = \coprod_i X_i$ then $H_0(X) = \bigoplus_i H_0(X_i)$ and $H^0(X) = \bigoplus_i H^0(X_i)$, $H_0(X) \sim H^0(X_i) \sim \mathbb{Z}$. Generator h defines path-connected component of x_0 satisfies following conditions:
   (a) h has infinite period
   (b) h is not divisible
   (c) If h_{i_0} generator of $H_0(X_{i_0})$ and h^i generator of $H^0(X_{i_0})$ then $h^{i_0} \frown h_{i_0} = h_{i_0}$ and $h^i \frown h_{i_0} = 0$ ($i_0 \neq i$).

Since we consider K^1 as analogue of H_1 then it is reasonable consider K^0 as analogue of H_0.

Definition 6.2. Let A be C^* algebra and $h \in K^0(A)$. A pair (A, h) is *noncommutative generalization of pointed space* if following conditions are hold:

1. h has infinite period
2. h is not divisible
3. if $A = \bigoplus_i A_i$ and $1_A = \sum_i 1_{A_i}$ then there exits such single index i_0 that $h \cdot [1_{A_{i_0}}] = 1_{KK_0(\mathbb{C},\mathbb{C})}$ and $h \cdot [1_{A_i}] = 0$.

6.3 Hurewicz homomorphism with respect to covering

Hurewicz homomorphism is in general homomorphism from noncommutative group to commutative one. So it can be decomposed by following way:

$$\pi_1(X) \to \pi_{ab}(X) \to H_1(X),$$

where $\pi_{ab}(X)$ is Abelian group defined as $\pi_{ab}(X) = \pi_1(X)/[\pi_1(X), \pi_1(X)]$
Algebraic topology has good notion of fundamental group. However good noncommutative generalization of fundamental group is not known. But every covering $\widetilde{X} \to X$ defines covering group $G(\widetilde{X}, X)$ which is factorgroup of fundamental group. If this group has natural structure of subgroup then one can define natural homomorphism $G(\widetilde{X}, X) \to H_1(X)$. Since $H_1(X)$ is Abelian we can take into account Abelian coverings only (see section 4) i.e. coverings with Abelian covering group. Abelian group is simultaneously subgroup and factorgroup if is direct summand. So if $G(\widetilde{X}, X)$ is direct summand of $\pi_{ab}(X)$ then it is natural homomorphism $G(\widetilde{X}, X) \to H_1(X)$.

Definition 6.3. Let $\pi : \widetilde{X} \to X$ be Abelian covering and $G(\widetilde{X}, X)$ is direct summand of $\pi_{ab}(X)$. Natural homomorphism $G(\widetilde{X}, X) \to H_1(X)$ is a *Hurewicz homomorphism with respect to π*.

Let us generalize this definition. Fundamental group is not defined for noncommutative C^* - algebras. However if $G(\widetilde{X}, X)$ is direct summand of $\pi_{ab}(X)$ is also direct summand for all intermediate subgroup G i.e. $G(\widetilde{X}, X) \subset G \subset \pi_{ab}(X)$.

This observation enable us define generalization of Hurewicz homomorphism with respect to covering by following way.

Definition 6.4. Let $\pi : A \to B$ be such Abelian covering of C^* - algebras that for all Abelian coverings $B \to C$ group $G(B, A)$ is direct summand of $G(C, A)$.*Hurewicz homomorphism with respect to* π $A \to B$ is natural homomorphism from $G(B, A)$ to $K^1(A)$.

Let us generalize this definition. Fundamental group is not defined for noncommutative C^* - algebras. However if $G(\widetilde{X}, X)$ is direct summand of $\pi_{ab}(X)$ is also direct summand for all intermediate subgroup G i.e. $G(\widetilde{X}, X) \subset G \subset \pi_{ab}(X)$.
This observation enable us define generalization of Hurewicz homomorphism with respect to covering by following way.

6.4 Noncommutative Hurewicz homomorphism

Noncommutative generalization of Hurewicz homomorphism is not group homomorphism. It is a set of homomorphism's condidions. In particular cases this condinions define unuque gruop homomorphism. In general this homomorphism does not exist and is not unique. Let G, H be finitely generated Abelian groups and $f : G \to H$ is group homomorphism. Let G_{tors} (resp. H_{tors} be torsion of G (resp. H). Then it is following commutative diagram with exact rows

$$\begin{CD}
0 @>>> G_{tors} @>>> G @>>> G/G_{tors} @>>> 0\\
@. @V{f_{tors}}VV @V{f}VV @VV{\overline{f}}V @.\\
0 @>>> H_{tors} @>>> H @>>> H/H_{tors} @>>> 0
\end{CD}$$

Homomorphism f uniquely defines both f_{tors} and $\overline{f}$, but not vice versa. So f_{tors} and $\overline{f}$ can be regarded as properties of f. If one of following conditions is satisfied

1. $G_{tors} \approx \{0\}$;
2. $G/G_{tors} \approx \{0\}$
3. $H_{tors} \approx \{0\}$
4. $H/H_{tors} \approx \{0\}$

then f_{tors} and $\overline{f}$ uniquely define f. Otherwise if A is C^* - algebra then it is following sequence:

$$\begin{CD}
0 @>>> \mathrm{Ext}^1_{\mathbb{Z}}(K_0(A), K_0(\mathbb{C})) @>>> KK^1(A, \mathbb{C}) @>>> \mathrm{Hom}(K^1(A), K_0(\mathbb{C})) @>>> 0\\
@. @V{\approx}VV @V{\approx}VV @VV{\approx}V @.\\
0 @>>> \mathrm{Rep}(K_0(A)_{tors}) @>>> K^1(A) @>>> \mathrm{Hom}(K^1(A), \mathbb{Z}) @>>> 0
\end{CD}$$

Construction of Hurewicz homomorphism generalization has properties of following diagrram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G_{tors} & \longrightarrow & G & \longrightarrow & \mathrm{Hom}(G/G_{tors}, \mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_{tors}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \overline{f}} & & \\
0 & \longrightarrow & \mathrm{Rep}(K^0(A)_{tors}) & \longrightarrow & K^1(A) & \longrightarrow & \mathrm{Hom}(K^1(A), \mathbb{Z}) & \longrightarrow & 0
\end{array}$$

where $G = G(B|A)$ is Abelian covering group and $\mathrm{Rep}(G)$ means representation group $\forall G$ (G is finite Abelian group) .
Rather we would like construct properties of f_{tors} and $\overline{f}$. First of all note that $K^*(A) \approx KK^*_G(\mathbb{C}, A)$. There are canonical pairings

$$G_{tors} \times \mathrm{Rep}(G_{tors}) \to \mathfrak{A},$$
$$\mathrm{Rep}(K^0(A)_{tors}) \times K^0(A_{tors}) \to \mathfrak{A}.$$

where $\mathfrak{A}$ is finite Abelian group.
So following pairing

$$\mathrm{Rep}(G_{tors}) \times K^0(A)_{tors} \to \mathfrak{A} \tag{35}$$

can be regarded as anlogue of isomorphism $G_{tors} \approx \mathrm{Rep}(K^0(A)_{tors})$. Kasparov intersection product $KK^*_G(\mathbb{C}, \mathbb{C}) \otimes KK^*_G(\mathbb{C}, B) \to KK^*_G(\mathbb{C}, B)$. From

$$KK^*_G(\mathbb{C}, \mathbb{C}) \approx \mathrm{Rep}(G_{tors}),$$

$$KK^0_G(\mathbb{C}, B) \approx KK^0(\mathbb{C}, A) \approx K^0(A).$$

it follows that it is following pairing.

$$\mathrm{Rep}(G_{tors}) \times K^0(A) \to K^0(A).$$

Since $\mathrm{Rep}(G_{tors})$ is finite above pairing does not depend on infinite part of $K^0(A)$ i.e.

$$\mathrm{Rep}(G_{tors}) \times K^0(A)_{tors} \to K^0(A)_{tors}.$$

Above formula is in fact pairing (35).
There are following natural pairings.

$$G/G_{tors} \times \mathrm{Hom}(G/G_{tors}, \mathbb{Z}) \to \mathbb{Z}.$$

$$K^1(A) \times \mathrm{Hom}(K^1(A), \mathbb{Z}) \to \mathbb{Z}.$$

So following pairing

$$K^1(A) \times \mathrm{Hom}(K^1(A), \mathbb{Z}) \to \mathfrak{A}.$$

can be regarded as analogue of isomorphism $G/G_{tors} \approx K^0(A)$ From Kasparov intersection product it follows next pairing

$$KK^1_G(\mathbb{C},\mathbb{C}) \times KK^1_G(\mathbb{C},B) \to KK^0_G(\mathbb{C},B) \tag{36}$$

If G is finitely generated Abelian group then

$$KK^1_G(\mathbb{C},\mathbb{C}) \approx \mathrm{Hom}(G,\mathbb{Z}) \approx \mathrm{Hom}(G/G_{tors},\mathbb{Z});$$
$$KK^1_G(B,\mathbb{C}) \approx KK^1(A,\mathbb{C}) \approx K^1(A).$$

6.5 Construction Hurewicz homomorphism generalization

Now we have all ingredients for construction of Hurewicz homomorphism generalization. Let (A,h) be noncommutative generalization of pointed space (see definition 6.2), $\pi : A \to B$ be Abelian covering which satisfies conditions of definition 6.4, and covering group $G = G(B,A)$ is finitely generated (Abelian) group.
Construction of Hurewicz homomorphism generalization with respect to π includes following steps.

1. It is natural isomorphism: $K^*_G(B) \to K^*(A)$;
2. It is natural isomorphism $G \sim KK^1_G(\mathbb{C},\mathbb{C})$;
3. $KK^1_G(\mathbb{C},\mathbb{C})$ acts on $KK^0_G(B,\mathbb{C}) \sim K^0_G(B) \sim K^0(A)$, So G acts on $K^0(A)$, it is pairing $G \times K^0(A) \to K^1(A)$;
4. Hurewicz homomorphism generalization with respect to π is defined as

$$G \ni g \mapsto (gh - h) \in K^1(A). \tag{37}$$

Definition 6.5. Let (A,h) be noncommutative generalization of pointed space and $\pi : A \to B$ be Abelian covering which satisfies conditions of definition 6.4, and covering group $G = G(B,A)$ is finitely generated (Abelian) group. An Abelian group homomorphism $\phi : G \to K^1(A)$ is called *Hurewicz homomorphism generalization with respect to* π if ϕ is defined by equation (37).

Remark 6.6. *Functionality of Hurewicz homomorphism*. Let $f : A \to B$ be *- homomorphism, $\overline{f}$ is homomorphism of fundamental groups with respect to $A \to \widetilde{A}$, $B \to \widetilde{B}$ coverings, (B,h), $(A, K^1(f)(h))$ generalizations of pointed spaces. Then following natural diagram

$$\begin{array}{ccc} G(\widetilde{B}|B) & \xrightarrow{\overline{f}} & G(\widetilde{A}|A) \\ \downarrow & & \downarrow \\ K^1(B) & \xrightarrow{K^1(f)} & K^1(B) \end{array}$$

is commutative. Vertical arrows of above diagram are generalizations of Hurewicz homomorphism defined by pairs (B,h), $(A, K^1(f)(h))$.

Let us remind universal coefficient theorem of KK theory

Theorem 6.7. *[16] Let A and B be separable C^* algebras with $A \in N$. Then there is a short exact sequence*

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B)) \xrightarrow{\delta} KK^*(A, B) \xrightarrow{\gamma} \mathrm{Hom}(K_*(A), K_*(B)) \to 0. \tag{38}$$

The map γ has degree 0 and δ has degree 1. The sequence is natural and splits unnaturally. So if $K_*(A)$ is divisible or $K_*(B)$ is divisible, then γ is isomorphism.

Particular case of theorem38 is following sequence:

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(K_0(A), \mathbb{Z}) \xrightarrow{\delta} K^1(A) \xrightarrow{\gamma} \mathrm{Hom}(K_1(A), \mathbb{Z}) \to 0.$$

Similarly if G is finitely generated Abelian group then it is following exact sequence:

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(KK^0_G(\mathbb{C}, \mathbb{C}), \mathbb{Z}) \xrightarrow{\alpha} KK^1_G(\mathbb{C}, \mathbb{C}) \xrightarrow{\beta} \mathrm{Hom}(KK^1_G(\mathbb{C}, \mathbb{C}), \mathbb{Z}) \to 0.$$

Generalization of Hurewicz homomorphism induces following natural homomorphisms between above exact sequences.

$$\begin{array}{ccccc}
\mathrm{Ext}^1_{\mathbb{Z}}(KK^0_G(\mathbb{C}, \mathbb{C}), \mathbb{Z})) & \xrightarrow{\alpha} & KK^1_G(\mathbb{C}, \mathbb{C}) & \xrightarrow{\beta} & \mathrm{Hom}(KK^1_G(\mathbb{C}, \mathbb{C}), \mathbb{Z}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Ext}^1_{\mathbb{Z}}(K_0(A), \mathbb{Z}) & \xrightarrow{\alpha} & K^1(A) & \xrightarrow{\beta} & \mathrm{Hom}(K_1(A), \mathbb{Z})
\end{array}$$

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