Noncommutative Generalization of Hurewicz homomorphism

Petr R. Ivankov

RPKB, Moscow, Russia
Let $X$ be topological space, $x_0 \in X$ and $\pi_n(X, x_0)$, $H_n(X)$ are $n$-th ($n \in \mathbb{N}$) and homotopy group of singular homology then there is following homomorphism:

$$\phi : \pi_n(X, x_0) \to H_n(X).$$

$\phi$ is named Hurewicz homomorphism.

Pair $(X, x_0)$ is named pointed space.

If $n = 1$ then homomorphism is defined by following way: Let $S^1$ be a circle then $H_1(S^1) \equiv \mathbb{Z}$. Let $c \in H_1(S^1)$ be generator of $H_1(S^1)$. Then Hurewicz homomorphism is defined by following expression:

If $[f] \in [S^1, s_0] = \pi_1(X, x_0)$ then $\phi([f]) = H_1(f)(c)$.

Let us try generalize Hurewicz homomorphism for $n = 1$ to noncommutative case.
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The **Gelfand - Naimark theorem** can be thought of as the construction of two contravariant functors (cofunctors for short) from the category of locally compact Hausdorff spaces to the category of $C^*$-algebras. The first cofunctor $C$ takes a compact space $X$ to the $C^*$-algebra $C(X)$ of continuous complex-valued functions on $X$, and takes a continuous map $f : X \to Y$ to its transpose $C(f) : C(Y) \to C(X)$ defined by following way: $C(f) = (h \mapsto hf); \ (h \in C(Y))$.

Otherwise there exists inverse functor $M$ that sets to any commutative $C^*$-algebra $A$ space of its characters $M(A)$. Many topological results related to locally compact spaces has its (noncommutative) algebraic analogues.
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The **Noncommutative geometry** is **THE POINT IS THAT THERE IS NO POINT**. Noncommutative $C^*$-algebra is being considered as noncommutative generalization of locally compact Hausdorff space.

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Noncommutative geometry contains MORE QUESTIONS THEN ANSWERS

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Noncommutative Generalization of Hurewicz homomorphism
Main questions

1. What is analogue of $H_1(X)$?
2. What is analogue of pointed space $(X, x_0)$?
3. What is analogue of $\pi_1(X, x_0)$?
4. What is analogue of Hurewicz homomorphism?

There is a set of versions of answers which depend on context. Analogue of $H_1(X)$ for Hurewicz theorem can be different from analogue of $H_1(X)$ for other problems.
There are three approaches for solution of the problem:

1. **Direct (Deductive)** From analogues of definitions to analogues of theorems;
2. **Inverse** From analogues of theorems to analogues of definitions;
3. **Combined** Simultaneous development of analogues of definitions and theorems.

Fundamental group notion

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One of theorem is analogue of **Hurewicz theorem**. Since Hurewicz theorem contains information about fundamental group the analogue of Hurewicz theorem could help to state definition of fundamental group.
1. Let $P$ be homotopy invariant covariant functor from Hausdorff locally compact topological spaces to Abelian groups which satisfy following condition $P(S^1) \sim \mathbb{Z}$; Then we can construct generalize of Hurewicz homomorphism replacing $H_1$ by $P$ and construct analogue of Hurewicz homomorphism. 

$$\phi : \pi_1(X, x_0) \to P(X);$$

2. Construction

Let $c$ be generator of Abelian group $P(S^1)$ and $f : (S, s_0) \to X$ continuous map which represent element $[f] \in \pi_1(X, x_0)$. We suppose that analogue of Hurewicz homomorphism $\phi$ is defined by following way:

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$\phi([f]) = P(f)(c)$;
Let $P$ be homotopy invariant covariant functor from Hausdorff locally compact topological spaces to Abelian groups which satisfy following condition $P(S^1) \sim \mathbb{Z}$; $M$ is contravariant functor from commutative $C^*$ algebras to Hausdorff locally compact topological spaces which sets for $C^*$ commutative algebra a space of its characters.

Then $PM$ is a contravariant homotopy invariant functor from from commutative $C^*$ - algebras to Abelian groups which satisfy condition $PM(C(S^1)) \sim \mathbb{Z}$

So noncommutative analogue of $P$ is a contravariant homotopy invariant functor $R$ from from (sub)category (noncommutative) $C^*$ - algebras to Abelian groups which satisfy condition $R(C(S^1)) \sim \mathbb{Z}$

Example Let $K^1$ be functor of $K$ homology. Then $K^1(C(S^1)) = \mathbb{Z}$. In this article the $K^1(A)$ as analogue of $H_1$ is being considered.
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Analogue of connected component

1. If $X = \bigsqcup_i X_i$ and all $X_i$ are connected then all algebras $C(X_i)$ are simple and $C(X) = \bigoplus_i C(X_i)$

2. If $C^*$ - algebra $A = \bigoplus_i A_i$ then $i$ - th connected component is associated to $A_i$ ($A_i$ is simple algebra).

3. If $A$ is unital then $1_A = \sum_i 1_{A_i}$ and $1_{A_i}$ is selfadjoint idempotent of $A$. 
Noncommutative analogue of pointed space \((X, x_0)\)

1. Any point \(x_0 \in X\) defines homomorphism \(H_0(\{x_0\}) \rightarrow H_0(X)\) and generator of \(h \in H_0(X)\). If \(X = \bigsqcup_i X_i\) then \(H_0(X) = \bigoplus_i H_0(X_i)\) and \(H^0(X) = \bigoplus_i H^0(X_i)\), \(H_0(X) \sim H^0(X_i) \sim \mathbb{Z}\). Generator \(h\) defines path-connected component of \(x_0\) satisfies following conditions:
   1. \(h\) has infinite period
   2. \(h\) is not divisible
   3. If \(h_{i_0}\) generator of \(H_0(X_{i_0})\) and \(h^i\) generator of \(H^0(X_{i_0})\) then \(h_{i_0} \sim h_{i_0} = h_{i_0}\) and \(h^i \sim h_{i_0} = 0\) \((i_0 \neq i)\).

2. Since we consider \(K^1\) as analogue of \(H_1\) then it is reasonable consider \(K^0\) as analogue of \(H_0\). If \(h' \in K^0(A)\) is analogue of \(h \in X\) then we can define following requirements.
   1. \(h'\) has infinite period
   2. \(h'\) is not divisible
   3. If \(A = \bigoplus_i A_i\) and \(1_A = \sum_i 1_{A_i}\) then there exits such single index \(i_0\) that \(h' \cdot [1_{A_{i_0}}] = 1_{KK_0(C, C)}\) and \(h' \cdot [1_{A_i}] = 0\).

Pair \((A, h')\) will be called analogue of pointed space.
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Main requirements to analogues of fundamental group and Hurewicz homomorphism

1 Analogue of fundamental group
   1 Definition Noncommutative analogue of fundamental group is a map which sets to any analogue of pointed space \((A, h)\) group \(\pi_1(A, h)\).
   2 Requirement If \(X\) is locally compact Hausdorff space then \(\pi_1(C(X), h) \sim \pi_1(X, x_0)\). If this requirement is satisfied then fundamental group \(\pi_1(A, h)\) is called ”good”.

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   1 Definition Noncommutative analogue of Hurewicz homomorphism is homomorphism from \(\pi_1(A, h)\) to \(K^1(A)\).
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Different definitions of fundamental group

1. **Definition 1** Let $X$ be a topological space and $x_0 \in X$ is its point. Then the fundamental group $\pi_1(X, x_0)$ as a set is a set of homotopy classes $[S^1, s_0; X, x_0]$. Since the noncommutative geometry is THE POINT IS THAT THERE IS NO POINT this definition is not suitable.

2. **Definition 2** Fundamental group is a group $G(\tilde{X}|X)$ of covering transformations of universal covering $\tilde{X}$ of $X$.

3. **Definition 1** does not have good noncommutative generalization. We need noncommutative analogue of $\tilde{X}$ for Definition 2 generalization. This problem is only partially solved.
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Analogue of $\tilde{X}$

1. Universal covering $\tilde{X}$ is universal (maximal) in category of coverings of $X$.

2. There are following approaches of definition of maximal covering object:
   - Define good analogue of category of coverings and looking for its universal object;
   - Define good analogue of covering objects and partial order on these objects. Then looking for maximal object and proving its unique property.
1 **Definition (Miyashita 1966)** Let $f : A \rightarrow B$ be homomorphism of algebras and $G$ is finite group of automorphisms of $A$. Let $h : A \otimes_B A \rightarrow \text{Map}(G, A)$ is a map defined by following way: $a_1 \otimes a_2 \mapsto (g \mapsto a_1 \otimes g a_2)$. Homomorphism $f$ is called $G$ - Galois if following two conditions are satisfied

1. $A = B^G$; ($G$ is denoted by $G(B|A)$).
2. Map $h$ is bijective.

2 If $A$ and $B$ is commutative then $M(B) \rightarrow M(A)$ is finitely listed covering.

3 If $f : A \rightarrow B$ is $G$ - Galois extension then there is natural isomorphism $K^*(A) \sim K^*_G(B)$

This definition provides good generalization of finitely listed covering. So we call *- homomorphism $f$ finitely listed covering if $f$ finite $G$ - Galois extension.
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1. Besides finite coverings we should have analogue of infinite coverings.

2. **Definition (Miyashita 1967)** Let $G$ be discrete group. Homomorphism $f : A \rightarrow B$ is called **locally finite $G$ - Galois extension** if there are fixed normal subgroups $N_{\lambda}$ ($\lambda \in \Lambda$) which satisfy the following conditions:
   
   1. $(G : N_{\lambda}) < \infty$ and $A \rightarrow B^{N_{\lambda}}$ is $G/N_{\lambda}$ - Galois extension;
   2. $B = \bigcup_{\lambda} B^{N_{\lambda}}$, and $\{B^{N_{\lambda}} : \lambda \in \Lambda\}$ is a directed set with respect to inclusion relation ($\bigcup_{\lambda} B^{N_{\lambda}}$ is directed union).

3. This definition do not provide good generalization of finitely listed covering by following reasons:
   
   1. Let $X$ be compact Hausdorff space. Suppose that $Y \rightarrow X$ infinitely listed covering and $G = G(Y|X)$ is infinite covering group;
   2. Then $Y$ is not compact and $C_{0}(Y)$ is not unital;
   3. If $A$ is unital algebra then every locally finite $G$ - Galois extension of $A$ is unital.
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**Noncommutative Generalization of Hurewicz homomorphism**
Noncommutative analogues of infinite coverings.

1 **Sketch of definition**

1. Let $A, B$ be $C^*$ algebras and $M(B)$ is algebra of multipliers of $B$.
2. **Locally finite covering** is locally finite $G$-Galois $^*$-homomorphism.

2 **Example**

3. Let $A$ be $C^*$ - algebra and $u \in A$ such unitary element that $[u] \in K_1(A)$ is nontrivial generator of infinite order.
4. It is (not unique) sequence of $C^*$ - algebras $A \subset A[v_1] \subset A[v_2] \subset \ldots$ which match following requirements: $v_n^n = u$ ($n \in \mathbb{N}$). $A(v_n)$ is $\mathbb{Z}_n$ - Galois extension.
5. It is evident left action of $C(u)$ on $A$. Let us define action of $C[u]$ on $C_0(\mathbb{R}$ by following way: $uf = e^{2\pi i x}f$ ($f \in C_0(\mathbb{R}$), $x \in \mathbb{R}$).
6. There are such algebra $B$ that $B \sim A \otimes_{C(u)} \mathbb{R}$ as $A[v_n] - \mathbb{R}$ as bimodules and there are $^*$ - homomorphisms $A(v_n) \rightarrow M(B)$.
Let $A$ be commutative $C^*$-algebra generated by two unitary elements $u$ and $v$: 
$uu^* = u^*u = vv^* = v^*v = 1; uv = vu$.

1. Let $B'$ be $C^*$-algebra generated by unitary elements $x', y'$: 
$x'x'^* = x'^*x' = y'y'^* = y'^*y' = 1; x'y' = y'x'$.

2. Let $f': A \rightarrow B' \mathbb{Z}_2$ be Galois $*$-homomorphism defined by following way: 
$u \mapsto x'^2; v \mapsto y'$.

3. Let $B'$ be $C^*$-algebra generated by unitary elements $x', y'$: 
$x''x''^* = x''^*x'' = y''y''^* = y''^*y'' = 1; x''y'' = -y''x''$.

4. Let $f': A \rightarrow B' \mathbb{Z}_2$ be Galois $*$-homomorphism defined by following way: 
$u \mapsto x''^2; v \mapsto y''$.

Morphisms $f'$ and $f''$ should be equivalent for good analogue of fundamental group. But $B' \sim B''$. So $*$-homomorphisms are not good covering morphisms.
**Definition**  \( f' : A \to B' \) and \( f'' : A \to B'' \) two coverings and \( G', G'' \) are their groups. Morphism of coverings is a pair \((\phi, B' \cdot H_{B''})\) of surjective group homomorphism \( \phi : G' \to G'' \) and \( B' - B'' \) bimodule \( B' \cdot H_{B''} \) which satisfies following conditions and \( KK \) compatibility axiom:

1. \( G' \) acts on \( B' \cdot H_{B''} \) this action makes \( B' \cdot H_{B''} \) equivariant left \( B' \) - module. This action and homomorphism \( \phi \) makes \( B' \cdot H_{B''} \) equivariant right \( B'' \) module;

2. \( B' \sim \text{End}_{B''}(B' \cdot H_{B''}) \) and \( B'' \sim \text{End}_{B'}(B' \cdot H_{B''})^{G'/G''} \)
Homomorphisms $f' : A \to B'$ and $f'' : A \to B''$ induce natural homomorphisms $KK_*(\cdot, A) \to KK_*(\cdot, B')$, $KK_*(\cdot, A) \to KK(\cdot, B'')$, $KK(B', \cdot) \to KK(A, \cdot)$, $KK(B', \cdot) \to KK(A, \cdot)$. Bimodule $B' H B''$ induces natural homomorphisms $KK(\cdot, B') \to KK(\cdot, B'')$ and $KK(B'', \cdot) \to KK(B'', \cdot)$.

**KK Compatibility axiom.** Following diagrams should be commutative:

\[
\begin{array}{ccc}
KK(B'', \cdot) & \longrightarrow & KK(B', \cdot) \\
\downarrow & & \downarrow \\
KK(A, \cdot) & & KK(A, \cdot)
\end{array}
\quad
\begin{array}{ccc}
KK(\cdot, B'') & \longleftarrow & KK(\cdot, B') \\
\downarrow & & \downarrow \\
KK(\cdot, A) & & KK(\cdot, A)
\end{array}
\]

**Question** For which $C^*$ algebras universal object of coverings category is exist?
Homomorphisms \( f' : A \to B' \) and \( f'' : A \to B'' \) induce natural homomorphisms \( KK_*(\cdot, A) \to KK_*(\cdot, B'), KK_*(\cdot, A) \to KK(\cdot, B''), KK(B', \cdot) \to KK(A, \cdot), KK(B', \cdot) \to KK(A, \cdot) \). Bimodule \( B' H_{B''} \) induces natural homomorphisms \( KK(\cdot, B') \to KK(\cdot, B'') \) and \( KK(B'', \cdot) \to KK(B'', \cdot) \).

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**Question** For which \( C^* \) algebras universal object of coverings category is exist?
1. Let homomorphisms $f' : A \rightarrow B'$ and $f'' : A \rightarrow B''$ are coverings. If $C$ is $C^*$- algebra than these coverings induce natural homomorphisms $f_C' : KK_*(C, B') \rightarrow KK_*(C, A)$ and $f_C'' : KK_*(C, B'') \rightarrow KK_*(C, A)$.

2. **Definition** We say that $f'$ is greater or equal than $f''$ ($f' \geq f''$) if for any $C^*$- algebra $C$ following condition is satisfied $\text{im}_C f_C' \subseteq f_C''$.

3. **Example** Let $X, X', X''$ be locally compact Hausdorff topological spaces and $f : X' \rightarrow X$, $g : X'' \rightarrow X$ are coverings. Then $fg : X'' \rightarrow X$ is covering and there are $*$- homomorphisms $C(f) : C(X') \rightarrow C(X)$ and $C(fg) : C(X'') \rightarrow C(X)$. In this case we have $C(fg) \geq C(f)$.

4. **Question** For which $C^*$ algebras do maximal covering exists?
1. Let homomorphisms $f' : A \to B'$ and $f'' : A \to B''$ are coverings. If $C$ is $C^*$- algebra than these coverings induce natural homomorphisms $f'^*_C : KK_*(C, B') \to KK_*(C, A)$ and $f''^*_C : KK_*(C, B'') \to KK_*(C, A)$.

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4. **Question** For which $C^*$ algebras do maximal covering exists?
1 Hurewicz homomorphism.
   1 Analogue of universal covering $\tilde{X}$ is not yet defined;
   2 We cannot define analogue of $\pi_1(X, x_0)$.
   3 So cannot define analogue analogue of Hurewicz homomorphism.

2 Hurewicz homomorphism associated with covering.
   1 Analogue of covering is defined;
   2 Analogue of covering group $G(Y|X)$ is $G(B|A)$;
   3 Let us define homomorphism $G(B|A) \to K^1(A)$. 
1 **Definition**

Abelian fundamental group $\pi_{ab}(X)$ is defined by following equation.

$$\pi_{ab}(X) = \pi_1(X, x_0)/[\pi_1(X, x_0), \pi_1(X, x_0)];$$

Note: Abelian fundamental group does not depend on $x_0$.

2 It is one to one correspondence between homomorphisms from $\pi_1(X)$ to Abelian group $A$ and homomorphisms from $\pi_{ab}(X)$ to $A$. Definition of $\pi_{ab}(X)$ could be easy then definition of $\pi_1(X)$

3 Since $K^1(A)$ is Abelian group then Hurewicz homomorphism $\pi_1(A)) \to K^1(A)$ could be decomposed by following way

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1. The universal covering is maximal among all coverings.

2. Since Hurewicz homomorphism depends on $\pi_{ab}(X)$ only class of coverings could be restricted to Abelian coverings (coverings with Abelian covering group).

3. Maximal Abelian covering is denoted by $X_{ab}$.

Class field theory considers Abelian extensions of fields only and provides particular calculation of Galois group. It is following analogy.

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Construction of Hurewicz homomorphism

1 Assumptions
   1 Let $A \to B$ be $G$-Galois extension and $G$ is finitely generated Abelian group;
   2 Let $(A, h)$ is analogue of pointed space.

2 Construction
   1 It is natural isomorphism: $K^*_G(B) \to K^*(A)$;
   2 It is natural isomorphism $G \simeq KK^1_G(\mathbb{C}, \mathbb{C})$;
   3 $KK^1_G(\mathbb{C}, \mathbb{C})$ acts on $KK^0_G(B, \mathbb{C}) \sim K^0_G(B) \sim K^0(A)$, So $G$ acts on $K^0(A)$, it is pairing $G \times K^0(A) \to K^1(A)$;
   4 Analogue of Hurewicz homomorphism is defined as $G \ni g \mapsto (gh - h) \in K^1(A)$.

This construction provides good Hurewicz homomorphism in particular cases.
Finite and infinite parts of Hurewicz homomorphism

Let $A \to B$ be covering with finitely generated Abelian covering group $G$ and $\text{Tors}(G)$ is its torsion. It is following diagram:

\[
\begin{array}{ccc}
\text{Tors}(G) & \to & G \\
\downarrow \phi^1_{\text{fin}} & & \downarrow \phi^1 \\
\text{Ext}^1(KK^0_G(\mathbb{C}, \mathbb{C}), \mathbb{Z}) & \to & KK^1_G(\mathbb{C}, \mathbb{C}) \\
\downarrow \phi^2_{\text{fin}} & & \downarrow \phi^2 \\
\text{Ext}^1(K_0(A), \mathbb{Z}) & \to & K^1(A) \\
\end{array}
\]

\[
\begin{array}{ccc}
G & \to & G/\text{Tors}(G) \\
\downarrow \phi^1 & & \downarrow \phi^1_{\text{inf}} \\
KK^1_G(\mathbb{C}, \mathbb{C}) & \to & \text{Hom}(KK^1_G(\mathbb{C}, \mathbb{C}), \mathbb{Z}) \\
\downarrow \phi^2 & & \downarrow \phi^2_{\text{inf}} \\
K^1(A) & \to & \text{Hom}(K^1(A), \mathbb{Z}) \\
\end{array}
\]

Here $\phi^2 \phi^1$ is Hurewicz homomorphism. Let us call $\phi_{\text{fin}} = \phi^2_{\text{fin}} \phi^1_{\text{fin}}$ ($\phi_{\text{inf}} = \phi^2_{\text{inf}} \phi^1_{\text{inf}}$) finite (infinite) part of Hurewicz homomorphism.

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Finite part of Hurewicz homomorphism

1. Let $A \to B$ be $G$-covering and $G \sim \mathbb{Z}_n$.
2. $B$ is finitely generated projective module. Let $r : K(A) \to K(A)$ is homomorphism defined by following way $K_0(A) \ni [P] \to [P \otimes_A B] \in K_0(A)$ Suppose that $\ker(r) = K \subseteq K_0(A)$ is isomorphic to $\mathbb{Z}_n$.
3. $G$ acts on $K$ and this action induce pairing: $G \times K \to \mathbb{Z}_n$;
4. It is subgroup $L \in K^1(A)$ that it is following covering: $L \times K \to \mathbb{Z}_n$;
5. These pairings induce homomorphism $G \to L$ which is a part of Hurewicz homomorphism $G \to L \subseteq K^1(A)$. 

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Noncommutative Generalization of Hurewicz homomorphism

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Infinite part of Hurewicz homomorphism

1. Suppose that $u_1, \ldots, u_n \in A$ such unitary elements that $[u_1], \ldots, [u_n] \in K_1(A)$ have infinite rank end these elements are not divisible in $K_1(A)$. Suppose that $[u_1], \ldots, [u_n]$ generate subgroup $\mathbb{Z}^n$.

2. It is $\mathbb{Z}^n$ covering $f : A \rightarrow B$ that $f_*([u_i]) = 0 \in K_1(B)$.

3. There are such generators $g_1, \ldots, g_n \in \mathbb{Z}^n \subseteq K_1(A)$ that $[u_i]\phi(g_j) \in \delta_{ij}1_{KK^0(\mathbb{C},\mathbb{C})}$.

4. Let $A$ is commutative then:
   1. $[u_1], \ldots, [u_n]$ are represented by continuous maps $r_i : M(A) \rightarrow S^1$ ($i = 1, \ldots n$).
   2. there are right inverse maps $s_i : S^1 \rightarrow M(A)$ of $r_i$ ($r_is_i = Id_{S^1}$).
   3. Hurewicz homomorphism is following map $g_i \mapsto r_i(c)$ where $c \in K^1(C(S^1))$ is generator of $K^1(C(S^1))$. 

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Noncommutative Generalization of Hurewicz homomorphism
Wait following results.

Thank You!